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# Tutorial 6:
# DERIVATION OF THE TRACTION KERNELS I: ELASTICITY EQUATIONS

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## Summary and objectives

In previous tutorials, we discussed different methods for deriving the displacement fundamental solutions for some types of operators. In this tutorial, detailed steps for the derivation of the traction kernels are discussed. The traction kernels for the two- and three-dimensional elasticity equations are considered herein; whereas, in the next tutorial we will discuss the derivation of the generalized traction kernels for the plate bending problem according to the Riessner theory.

## 1 The direction of the fundamental load

Recall tutorial 3, the stress-displacement relationships can be written as follows:

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}u_{\theta,\theta}\delta_{ij} + \frac{u_{i,j}+u_{j,i}}{2}\right) \tag{1}$$

Such equations related the stresses to the displacements at a certain point **x**; therefore equation (1) can be re-written as follows:

$$\sigma_{ij}(\mathrm{x}) = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}u_{\theta,\theta}(\mathrm{x})\delta_{ij} + \frac{u_{i,j}(\mathrm{x})+u_{j,i}(\mathrm{x})}{2}\right) \tag{2}$$

If the same equation will be written to represent the same relationships for the fundamental field, a new index *j* will be introduced to represent the direction of the load at the point **ξ**:

$$\sigma_{kij}(\xi,\mathrm{x}) = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}U_{k\theta,\theta}(\xi,\mathrm{x})\delta_{ij} + \frac{U_{ki,j}(\xi,\mathrm{x})+U_{kj,i}(\xi,\mathrm{x})}{2}\right) \tag{3}$$

The following notes should be emphasized:

1- We introduced the direction of the load as the first index to follow the convention used in Primer 3; however it is arbitrary.
2- It is preferred to introduce the direction of the load as the first index to follow the notation of the two-point kernel between **ξ** and **x**.
3- Provided that the stresses in equation (3) represent the fundamental field, the corresponding displacements will be the displacement fundamental solution fields.
4- The distributed loading or any other type of body or domain loading will be not considered in the fundamental state as the applied source load will be the only considered loading.

Equation (3) will be the basic equation to derive the traction fundamental solution kernels.

## 2 The two-dimensional problem

In the two-dimensional case, the following expression for the displacement fundamental kernels can be obtained as follows [1]:

$$U_{ki} = \frac{1}{8\pi G(1-\nu)}\left[-(3-4\nu)\ln r\,\delta_{ki} + r_{,i}r_{,k}\right] \tag{4}$$

Taking the derivatives with respect to the field point (recall tutorial 2), we can obtain the following derivatives:

$$U_{ki,j} = \frac{1}{8\pi G(1-\nu)}\left\{-(3-4\nu)\delta_{ki}\frac{1}{r}r_{,j} + \left(r_{,i}\frac{\delta_{kj}-r_{,k}r_{,j}}{r} + r_{,k}\frac{\delta_{ij}-r_{,i}r_{,j}}{r}\right)\right\} \tag{5}$$

$$= \frac{1}{8\pi G(1-\nu)r}\left\{-(3-4\nu)\delta_{ki}r_{,j} + r_{,i}\delta_{kj} + r_{,k}\delta_{ij} - 2r_{,i}r_{,j}r_{,k}\right\} \tag{6}$$

Interchanging the indices, we get:

$$U_{kj,i} = \frac{1}{8\pi G(1-\nu)r}\left\{-(3-4\nu)\delta_{kj}r_{,i} + r_{,j}\delta_{ki} + r_{,k}\delta_{ij} - 2r_{,i}r_{,j}r_{,k}\right\} \tag{7}$$

Putting *i=j* (as a dummy index), we get:

$$U_{k\theta,\theta} = \frac{1}{8\pi G(1-\nu)r}\left\{-(3-4\nu)r_{,k} + r_{,k} + 2r_{,k} - 2r_{,k}\right\} \tag{8}$$

$$= \frac{1}{8\pi G(1-\nu)r}\left\{-2(1-2\nu)r_{,k}\right\} \tag{9}$$

Then:

$$\frac{U_{ki,j}+U_{kj,i}}{2}=\frac{1}{8\pi G(1-\nu)r}\left\{-2r_{,i}r_{,j}r_{,k}+r_{,k}\delta_{ij}-(1-2\nu)\left(r_{,i}\delta_{kj}+r_{,j}\delta_{ki}\right)\right\} \quad (10)$$

From equation (3), we get:

$$\sigma_{kij}=\frac{E}{1+\nu}\frac{1}{8\pi G(1-\nu)r}\times$$

$$\left\{\frac{\nu}{1-2\nu}\left(-2(1-2\nu)r_{,k}\right)\delta_{ij}-2r_{,i}r_{,j}r_{,k}+r_{,k}\delta_{ij}-(1-2\nu)\left(r_{,i}\delta_{kj}+r_{,j}\delta_{ki}\right)\right\} \quad (11)$$

$$=\frac{-1}{4\pi G(1-\nu)r}\left\{-2\nu\delta_{ij}r_{,k}+r_{,k}\delta_{ij}-2r_{,i}r_{,j}r_{,k}-(1-2\nu)\left(r_{,i}\delta_{kj}+r_{,j}\delta_{ki}\right)\right\} \quad (12)$$

$$=\frac{-1}{4\pi G(1-\nu)r}\left\{-(1-2\nu)r_{,k}\delta_{ij}+2r_{,i}r_{,j}r_{,k}+(1-2\nu)\left(r_{,i}\delta_{kj}+r_{,j}\delta_{ki}\right)\right\} \quad (13)$$

Recall, the traction-stress relationship [1]:

$$T_{ki}=\sigma_{kij}n_j \quad (14)$$

or:

$$T_{ki}=\frac{-1}{4\pi G(1-\nu)r}\left\{-(1-2\nu)r_{,k}n_i+2r_{,i}r_{,n}r_{,k}+(1-2\nu)\left(r_{,i}n_k+r_{,n}\delta_{ki}\right)\right\} \quad (15)$$

$$=\frac{-1}{4\pi G(1-\nu)r}\left\{r_{,n}\left[(1-2\nu)\delta_{ki}+2r_{,i}r_{,k}\right]+(1-2\nu)\left(r_{,i}n_k-r_{,k}n_i\right)\right\} \quad (16)$$

Equation (16), represents the final form of the traction fundamental solution kernels for the two-dimensional case.

## 3 The three-dimensional problem

The same steps can be followed to obtain the final form for the traction fundamental solution kernels for the three-dimensional case:

$$U_{ki}=\frac{1}{16\pi G(1-\nu)r}\left[(3-4\nu)\delta_{ki}+r_{,i}r_{,k}\right] \quad (17)$$

$$U_{ki,j}=\frac{1}{16\pi G(1-\nu)}\left\{\left[(3-4\nu)\delta_{ki}+r_{,i}r_{,k}\right]\left(\frac{-1}{r^2}\right)r_{,j}+\frac{1}{r}\left[0+\left(r_{,i}\frac{\delta_{kj}-r_{,k}r_{,j}}{r}+r_{,k}\frac{\delta_{ij}-r_{,i}r_{,j}}{r}\right)\right]\right\} \quad (18)$$

$$= \frac{1}{16\pi G(1-\nu)r^2}\left\{-(3-4\nu)\delta_{ki}r_{,j} - r_{,i}r_{,k}r_{,j} + \delta_{kj}r_{,i} + \delta_{ij}r_{,k} - 2r_{,i}r_{,k}r_{,j}\right\} \quad (19)$$

$$= \frac{-1}{16\pi G(1-\nu)r^2}\left\{(3-4\nu)\delta_{ki}r_{,j} - \delta_{kj}r_{,i} - \delta_{ij}r_{,k} + 3r_{,i}r_{,k}r_{,j}\right\} \quad (20)$$

$$U_{kj,i} = \frac{-1}{16\pi G(1-\nu)r^2}\left\{(3-4\nu)\delta_{kj}r_{,i} - \delta_{ki}r_{,j} - \delta_{ij}r_{,k} + 3r_{,i}r_{,k}r_{,j}\right\} \quad (21)$$

$$U_{k\theta,\theta} = \frac{-1}{16\pi G(1-\nu)r^2}\left\{(3-4\nu)r_{,k} - r_{,k} - 3r_{,k} + 3r_{,k}\right\} \quad (22)$$

$$= \frac{-1}{16\pi G(1-\nu)r^2}2(1-2\nu)r_{,k} \quad (23)$$

$$\frac{U_{ki,j} + U_{kj,i}}{2} = \frac{-1}{16\pi G(1-\nu)r^2}\left\{3r_{,i}r_{,k}r_{,j} - \delta_{ij}r_{,k} + (1-2\nu)\left(\delta_{ki}r_{,j} + \delta_{kj}r_{,i}\right)\right\} \quad (24)$$

$$\sigma_{kij} = \frac{E}{1+\nu}\frac{(-1)}{16\pi G(1-\nu)r^2}\times$$

$$\left\{3r_{,i}r_{,k}r_{,j} - \delta_{ij}r_{,k} + (1-2\nu)\left(\delta_{ki}r_{,j} + \delta_{kj}r_{,i}\right) + \frac{\nu}{1-2\nu}\left(2(1-2\nu)r_{,k}\right)\delta_{ij}\right\} \quad (25)$$

$$= \frac{-1}{8\pi(1-\nu)r^2}\left\{3r_{,i}r_{,k}r_{,j} - (1-2\nu)\delta_{ij}r_{,k} + (1-2\nu)\left(\delta_{ki}r_{,j} + \delta_{kj}r_{,i}\right)\right\} \quad (26)$$

$$T_{ki} = \frac{-1}{8\pi(1-\nu)r^2}\left\{3r_{,i}r_{,k}r_{,n} + (1-2\nu)\left(-r_{,k}n_i + \delta_{ki}r_{,n} + n_k r_{,i}\right)\right\} \quad (27)$$

$$= \frac{-1}{8\pi(1-\nu)r^2}\left\{r_{,n}\left[(1-2\nu)\delta_{ki} + 3r_{,i}r_{,k}\right] + (1-2\nu)\left[n_k r_{,i} - r_{,k}n_i\right]\right\} \quad (28)$$

## 3 Summary and conclusions

In this tutorial we discussed in details the algebraic steps for obtaining the traction fundamental solution kernels for the two- and the three-dimensional elasticity equations. In the next tutorial we will consider the same derivation for plate bending problems.

## References and Further Reading

[1] Brebbia, C.A. & Dominguez, J., Boundary Elements: An Introductory Course, WIT Press, Southampton, UK., McGraw Hill, 1992.