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Heritage and early history of the boundary element method

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Abstract

This article explores the rich heritage of the boundary element method (BEM) by examining its mathematical foundation from the potential theory, boundary value problems, Green's functions, Green's identities, to Fredholm integral equations. The 18th to 20th century mathematicians, whose contributions were key to the theoretical development, are honored with short biographies. The origin of the numerical implementation of boundary integral equations can be traced to the 1960s, when the electronic computers had become available. The full emergence of the numerical technique known as the boundary element method occurred in the late 1970s. This article reviews the early history of the boundary element method up to the late 1970s.

Keywords: Boundary element method; Green's functions; Green's identities; Boundary integral equation method; Integral equation; History

1. Introduction

After three decades of development, the boundary element method (BEM) has found a firm footing in the arena of numerical methods for partial differential equations. Comparing to the more popular numerical methods, such as the Finite Element Method (FEM) and the Finite Difference Method (FDM), which can be classified as the *domain* methods, the BEM distinguish itself as a *boundary* method, meaning that the numerical discretization is conducted at reduced spatial dimension. For example, for problems in three spatial dimensions, the discretization is performed on the bounding surface only; and in two spatial dimensions, the discretization is on the boundary contour only. This reduced dimension leads to smaller linear systems, less computer memory requirements, and more efficient computation. This effect is most pronounced when the domain is unbounded. Unbounded domain needs to be truncated and approximated in domain methods. The BEM, on the other hand, automatically models the behavior at infinity without the need of deploying a mesh to approximate it. In the modern day industrial settings, mesh preparation is the most labor intensive and the most costly portion in numerical modeling, particularly for the FEM [9] Without the need of dealing with the interior mesh, the BEM is more cost effective in mesh preparation. For problems involving moving boundaries, the adjustment of the mesh is much easier with the BEM; hence it is again the preferred tool. With these advantages, the BEM is indeed an essential part in the repertoire of the modern day computational tools.

In order to gain an objective assessment of the success of the BEM, as compared to other numerical methods, a search is conducted using the *Web of Science*[SM], an online bibliographic database. Based on the keyword search, the total number of journal publications found in the *Science Citation Index Expanded* [195] was compiled for several numerical methods. The detail of the search technique is described in Appendix. The result, as summarized in Table 1, clearly indicates that the finite element method (FEM) is the most popular with more than 66,000 entries. The finite difference method (FDM) is a distant second with more than 19,000 entries, less than one third of the FEM. The BEM ranks third with more than 10,000 entries, more than one half of the FDM. All other methods, such as the finite volume method (FVM) and the collocation method (CM), trail far behind. Based on this bibliographic search,

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Table 1
Bibliographic database search based on the Web of Science

Numerical method	Search phrase in topic field	No. of entries
FEM	‘Finite element’ or ‘finite elements’	66,237
FDM	‘Finite difference’ or ‘finite differences’	19,531
BEM	‘Boundary element’ or ‘boundary elements’ or ‘boundary integral’	10,126
FVM	‘Finite volume method’ or ‘finite volume methods’	1695
CM	‘Collocation method’ or ‘collocation methods’	1615

Refer to Appendix A for search criteria. (Search date: May 3, 2004).

we can conclude that the popularity and versatility of BEM falls behind the two major methods, FEM and FDM. However, BEM’s leading role as a specialized and alternative method to these two, as compared to all other numerical methods for partial differential equations, is unchallenged.

Fig. 1 presents the histogram of the number of journal papers published annually, containing BEM as a keyword. It shows that the growth of BEM literature roughly follows the S-curve pattern predicted by the theory of technology diffusion [75]. Based on the data, we observe that after the ‘invention of the technology’ in the late 1960s and early 1970s, the number of published literature was very small; but it was on an exponential growth rate, until it reached an inflection point around 1991. After that time, the annual publication continued to grow, but at a decreasing rate. A sign of a technology reaching its maturity is marked by the leveling off of its production. Although it might be too early to tell, there is an indication that the number of annual BEM publications is reaching a steady state at about 700–800 papers per year. For comparison, this number for the FEM is about 5000 articles per year, and for the FDM, it is about 1400.

As the BEM is on its way to maturity, it is of interest to visit its history. Although there exist certain efforts toward the writing of the history of the FEM [84,127] and the FDM [131,193], relatively little has been done for the BEM. The present article is aimed at taking a first step toward the construction of a history for the BEM.

Before reviewing its modern development, we shall first explore the rich heritage of the BEM, particularly its mathematical foundation from the 18th century to the early 20th. The historical development of the potential theory, Green’s function, and integral equations are reviewed. To interest the beginners of the field, biographical sketches celebrating the pioneers, whose contributions were key to the mathematical foundation of the BEM, are provided. The coverage continues into the first half of the 20th century, when early numerical efforts were attempted even before the electronic computers were invented.

Numerical methods cannot truly prosper until the invention and then the wide availability of the electronic computers in the early 1960s. It is of little surprise that both the FEM and the BEM started around that time. For the BEM, multiple efforts started around 1962. A turning point that launched a series of connected efforts, which soon developed into a movement, can be traced to 1967. In the 1970s, the BEM was still a novice numerical technique, but saw an exponential growth. By the end of it, textbooks were

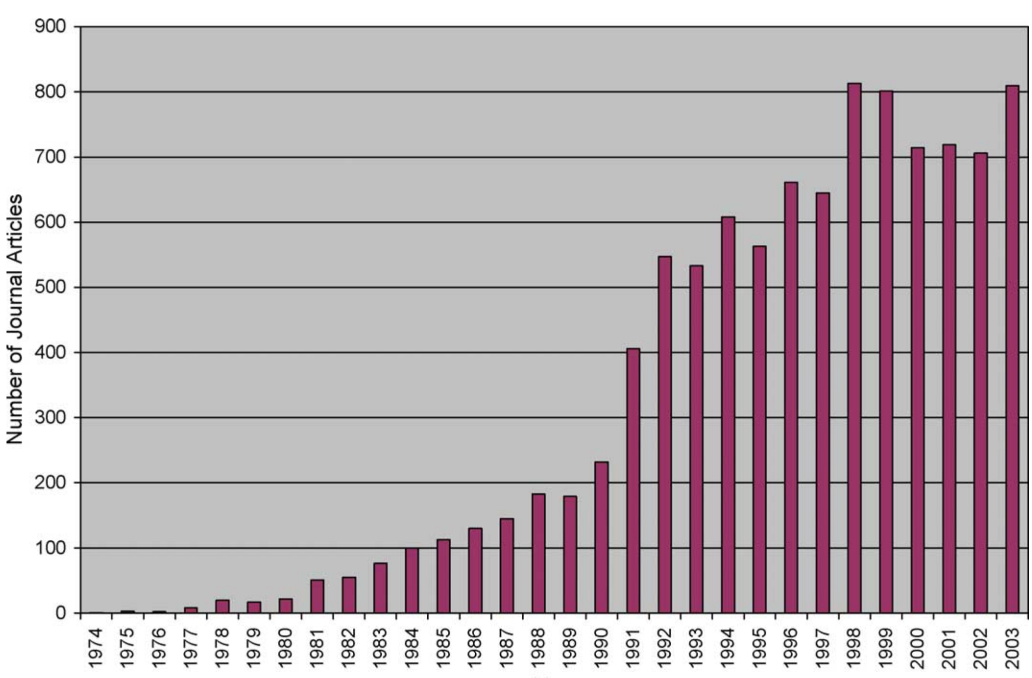


Fig. 1. Number of journal articles published by the year on the subject of BEM, based on the *Web of Science* search. Refer to Appendix for the search criteria. (Search date: May 3, 2004).

written and conferences were organized on BEM. This article reviews the early development up to the late 1970s, leaving the latter development to future writers.

Before starting, we should clarify the use of the term 'boundary element method' in this article. In the narrowest view, one can argue that BEM refers to the numerical technique based on the method of weighted residuals, mirroring the finite element formulation, except that the weighing function used is the fundamental solution of governing equation in order to eliminate the need of domain discretization [19,21]. Or, one can view BEM as the numerical implementation of boundary integral equations based on Green's formula, in which the piecewise element concept of the FEM is utilized for the discretization [108]. Even more broadly, BEM has been used as a generic term for a variety of numerical methods that use a boundary or boundary-like discretization. These can include the general numerical implementation of boundary integral equations, known as the boundary integral equation method (BIEM) [54], whether elements are used in the discretization or not; or the method known as the indirect method that distributes singular solutions on the solution boundary; or the method of fundamental solutions in which the fundamental solutions are distributed outside the domain in discrete or continuous fashion with or without integral equation formulation; or even the Trefftz method which distribute non-singular solutions. These generic adoptions of the term are evident in the many articles appearing in the journal of *Engineering Analysis with Boundary Elements* and many contributions in the Boundary Element Method conferences. In fact, the theoretical developments of these methods are often intertwined. Hence, for the purpose of the current historical review, we take the broader view and consider into this category all numerical methods for partial differential equations in which a reduction in mesh dimension from a domain-type to a boundary-type is accomplished. More properly, these methods can be referred to as 'boundary methods' or 'mesh reduction methods.' But we shall yield to the popular adoption of the term 'boundary element method' for its wide recognition. It will be used interchangeably with the above terms.

2. Potential theory

The Laplace equation is one of the most widely used partial differential equations for modeling science and engineering problems. It typically comes from the physical consequence of combining a phenomenological gradient law (such as the Fourier law in heat conduction and the Darcy law in groundwater flow) with a conservation law (such as the heat energy conservation and the mass conservation of an incompressible material). For example, Fourier law was presented by *Jean Baptiste Joseph Fourier* (1768–1830) in 1822 [66]. It states that the heat flux in a thermal conducting medium is proportional to the spatial gradient of temperature distribution

$$\mathbf{q} = -k\nabla T \tag{1}$$

where $\mathbf{q}$ is the heat flux vector, k is the thermal conductivity, and T is the temperature. The steady state heat energy conservation requires that at any point in space the divergence of the flux equals to zero:

$$\nabla \cdot \mathbf{q} = 0 \tag{2}$$

Combining (1) and (2) and assuming that k is a constant, we obtain the Laplace equation

$$\nabla^2 T = 0 \tag{3}$$

For groundwater flow, similar procedure produces

$$\nabla^2 h = 0 \tag{4}$$

where h is the piezometric head. It is of interest to mention that the notation ∇ used in the above came form *William Rowan Hamilton* (1805–1865). The symbol ∇, known as 'nabla', is a Hebrew stringed instrument that has a triangular shape [73].

The above theories are based on physical quantities. A second way that the Laplace equation arises is through the mathematical concept of finding a 'potential' that has no direct physical meaning. In fluid mechanics, the velocity of an incompressible fluid flow satisfies the divergence equation

$$\nabla \cdot \mathbf{v} = 0 \tag{5}$$

which is again based on the mass conservation principle. For an inviscid fluid flow that is irrotational, its curl is equal to zero:

$$\nabla \times \mathbf{v} = 0 \tag{6}$$

It can be shown mathematically that the identity (6) guarantees the existence of a scalar potential ϕ such that

$$\mathbf{v} = \nabla \phi \tag{7}$$

Combining (5) and (7) we again obtain the Laplace equation. We notice that ϕ, called the velocity potential, is a mathematical conceptual construction; it is not associated with any measurable physical quantity. In fact, the phrase 'potential function' was coined by *George Green* (1793–1841) in his 1828 study [81] of electrostatics and magnetics: electric and magnetic potentials were used as convenient tools for manipulating the solution of electric and magnetic forces.

The original derivation of Laplace equation, however, was based on the study of gravitational attraction, following the third law of motion of *Isaac Newton* (1643–1727)

$$\mathbf{F} = -\frac{G m_1 m_2 \mathbf{r}}{|\mathbf{r}|^3} \tag{8}$$

where $\mathbf{F}$ is the force field, G is the gravitational constant, m_1 and m_2 are two concentrated masses, and $\mathbf{r}$ is the distance vector between the two masses. *Joseph-Louis Lagrange*

(1736–1813) in 1773 was the first to recognize the existence of a potential function that satisfied the above equation [111]

$$\phi = \frac{1}{r} \tag{9}$$

whose spatial gradient gave the gravity force field

$$\mathbf{F} = G m_1 m_2 \nabla \phi \tag{10}$$

Subsequently, *Pierre-Simon Laplace* (1749–1827) in his study of celestial mechanics demonstrated that the gravity potential satisfies the Laplace equation. The equation was first presented in polar coordinates in 1782, and then in the Cartesian form in 1787 as [109]:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{11}$$

The Laplace equation, however, had been used earlier in the context of hydrodynamics by *Leonhard Euler* (1707–1783) in 1755 [63], and by Lagrange in 1760 [110]. But Laplace was credited for making it a standard part of mathematical physics [15,100]. We note that the gravity potential (9) satisfying (11) represents a concentrated mass. Hence it is a 'fundamental solution' of the Laplace equation.

Simeon-Denis Poisson (1781–1840) derived in 1813 [132] the equation of force potential for points interior to a body with mass density ρ as

$$\nabla^2 \phi = -4\pi\rho \tag{12}$$

This is known as the Poisson equation.

2.1. Euler

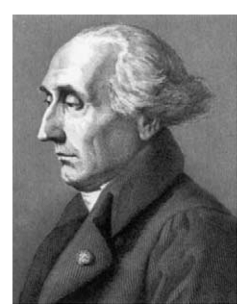
Leonhard Euler (1707–1783) was the son of a Lutheran pastor who lived near Basel, Switzerland. While studying theology at the University of Basel, Euler was attracted to mathematics by the leading mathematician at the time, *Johann Bernoulli* (1667–1748), and his two mathematician sons, *Nicolaus* (1695–1726) and *Daniel* (1700–1782). With no opportunity in finding a position in Switzerland due to his young age, Euler followed Nicolaus and Daniel to Russia. Later, at the age of 26, he succeeded Daniel as the chief mathematician of the Academy of St Petersburg. Euler surprised the Russian mathematicians by computing in 3 days some astronomical tables whose construction was expected to take several months.

In 1741 Euler accepted the invitation of Frederick the Great to direct the mathematical division of the Berlin Academy, where he stayed for 25 years. The relation with the King, however, deteriorated toward the end of his stay; hence Euler returned to St Petersburg in 1766. Euler soon became totally blind after returning to Russia. By dictation, he published nearly half of all his papers in the last 17 years of his life. In his words, '*Now I will have less distraction*'. Without doubt, Euler was the most prolific and versatile scientific writer of all times. During his lifetime, he published more than 700 books and papers, and it took St Petersburg's Academy the next 47 years to publish the manuscripts he left behind [31]. The modern effort of publishing Euler's collected works, the *Opera Omnia* [64] begun in 1911. However, after 73 volumes and 25,000 pages, the work is unfinished to the present day.

Euler contributed to many branches of mathematics, mechanics, and physics, including algebra, trigonometry, analytical geometry, calculus, complex variables, number theory, combinatorics, hydrodynamics, and elasticity. He was the one who set mathematics into the modern notations. We owe Euler the notations of 'e' for the base of natural logs, 'π' for pi, 'i' for $\sqrt{-1}$, '$\sum$' for summation, and the concept of functions.

Carl Friedrich Gauss (1777–1855) has been called the greatest mathematician in modern mathematics for his setting up the rigorous foundation for mathematics. Euler, on the other hand, was more intuitive and has been criticized by pure mathematicians as being lacking rigor. However, by the number indelible marks that Euler left in many science and engineering fields, he certainly earned the title of the greatest *applied* mathematician ever lived [58].

2.2. Lagrange



Joseph-Louis Lagrange (1736–1813), Italian by birth, German by adoption, and French by choice, was next to Euler the foremost mathematician of the 18th century. At age 18 he was appointed Professor of Geometry at the Royal Artillery School in Turin. Euler was impressed by his work, and arranged a prestigious position for him in Prussia. Despite the inferior condition in Turin, Lagrange only wanted to be able to devote his time to mathematics; hence declined the offer. However, in 1766, when Euler left Berlin for St Petersburg, Frederick the Great arranged for Lagrange to fill the vacated post. Accompanying the invitation was a modest message saying, '*It is necessary that the greatest* *geometer of Europe should live near the greatest of Kings.*' To D'Alembert, who recommended Lagrange, the king wrote, '*To your care and recommendation am I indebted for having replaced a half-blind mathematician with a mathematician with both eyes, which will especially please the anatomical members of my academy*.'

After the death of Frederick, the situation in Prussia became unpleasant for Lagrange. He left Berlin in 1787 to become a member of the Académie des Sciences in Paris, where he remained for the rest of his career. Lagrange's contributions were mostly in the theoretical branch of mathematics. In 1788 he published the monumental work *Mécanique Analytique* that unified the knowledge of mechanics up to that time. He banished the geometric idea and introduced differential equations. In the preface, he proudly announced: '*One will not find figures in this work. The methods that I expound require neither constructions, nor geometrical or mechanical arguments, but only algebraic operations, subject to a regular and uniform course*.' [31,128].

2.3. Laplace

Pierre-Simon Laplace (1749–1827), born in Normandy, France, came from relatively humble origins. But with the help of *Jean le Rond D'Alembert* (1717–1783), he was appointed Professor of Mathematics at the Paris Ecole Militaire when he was only 20-year old. Some years later, as examiner of the scholars of the royal artillery corps, Laplace happened to examine a 16-year old sub-lieutenant named Napoleon Bonaparte. Fortunately for both of their careers, the examinee passed. When Napoleon came to power, Laplace was rewarded: he was appointed the Minister of Interior for a short period of time, and later the President of the Senate.

Among Laplace's greatest achievement was the five-volume *Traité du Mécanique Céleste* that incorporated all the important discoveries of planetary system of the previous century, deduced from Newton's law of gravitation. Upon presenting the monumental work to Napoleon, the emperor teasingly chided Laplace for an apparent oversight: 'They told me that you have written this huge book on the system of the universe without once mentioning its Creator'. Whereupon Laplace drew himself up and bluntly replied, 'I have no need for that hypothesis.' [31].

He was eulogized by his disciple Poisson as 'the Newton of France' [86]. Among the important contributions of Laplace in mathematics and physics included probability, Laplace transform, celestial mechanics, the velocity of sound, and capillary action. He was considered more than anyone else to have set the foundation of the probability theory [76].

2.4. Fourier

Jean Baptiste Joseph Fourier (1768–1830), born in Auxerre, France, was the ninth of the 12 children of his father's second marriage. One of his letters showed that he really wanted to make a major impact in mathematics: 'Yesterday was my 21st birthday; at that age Newton and Pascal had already acquired many claims to immortality'. In 1790 Fourier became a teacher at the Benedictine College, where he had studied earlier. Soon after, he was entangled in the French Revolution and joined the local revolutionary committee. He was arrested in 1794, and almost went to the guillotine. Only the political changes resulted in his being released. In 1794 Fourier was admitted to the newly established Ecole Normale in Paris, where he was taught by Lagrange, Laplace, and *Gaspard Monge* (1746–1818). In 1797 he succeeded Lagrange in being appointed to the Chair of Analysis and Mechanics.

In 1978, Fourier joined Napoleon's army in its invasion of Egypt as a scientific advisor. It was there that he recorded many observations that later led to his work in heat diffusion. Fourier returned to Paris in 1801. Soon Napoleon appointed him as the Prefect of Isére, headquartered at Grenoble. Among his achievements in this administrative position included the draining of swamps of Bourgoin and the construction of a new highway between Grenoble and Turin. Some of his most important scientific contributions came during this period (1802–1814). In 1807 he completed his memoir *On the Propagation of Heat in Solid Bodies* in which he not only expounded his idea about heat diffusion, but also outlined his new method of mathematical analysis, which we now call Fourier analysis. This memoir, however, was never published, because one of its examiner, Lagrange, objected to his use of trigonometric series to express initial temperature. Fourier was elected to the Académie des Sciences in 1817. In 1822 he published *The Analytical Theory of Heat* [66], 10 years after its winning the Institut de France competition of the Grand Prize in Mathematics in 1812. The judges, however, criticized that he had not proven the completeness of the trigonometric (Fourier) series. The proof would come years later by *Johann Peter Gustav Lejeune Dirichlet* (1805–1859) [80].

2.5. Poisson

Simeon-Denis Poisson (1781–1840) was born in Pithiviers, France. In 1796 Poisson was sent to Fontainebleau to enroll in the Ecole Centrale. He soon showed great talents for learning, especially mathematics. His teachers at the Ecole Centrale were highly impressed and encouraged him to sit in the entrance examinations for the Ecole Polytechnique in Paris, the premiere institution at the time. Although he had far less formal education than most of the students taking the examinations, he achieved the top place. His teachers Laplace and Lagrange quickly saw his mathematical talents and they became friends for life. In his final year of study he wrote a paper on the theory of equations and Bézout's theorem, and this was of such quality that he was allowed to graduate in 1800 without taking the final examination. He proceeded immediately to the position equivalent to the present-day Assistant Professor in the Ecole Polytechnique at the age of 19, mainly on the strong recommendation of Laplace. It was quite unusual for anyone to gain their first appointment in Paris, as most of the top mathematicians had to serve in the provinces before returning to Paris. Poisson was named Associate Professor in 1802, and Professor in 1806 to fill the position vacated by Fourier when he was sent by Napoleon to Grenoble. In 1813 in his effort to answer the challenge question for the election to the Académie des Sciences, he developed the Poisson eqaution (12) to solve the electrical field caused by distributed electrical charges in a body.

Poisson made great contributions in both mathematics and physics. His name is attached to a wide variety of ideas, for example, Poisson's integral, Poisson equation, Poisson brackets in differential equations, Poisson's ratio in elasticity, and Poisson's constant in electricity [128].

2.6. Hamilton

William Rowan Hamilton (1805–1865) was a precocious child. At the age of 5, he read Greek, Hebrew, and Latin; at 10, he was acquainted with half a dozen of oriental languages. He entered Trinity College, Dublin at the age of 18. His performance was so outstanding that he was appointed Professor of Astronomy and the Royal Astronomer of Ireland when he was still an undergraduate at Trinity. Hamilton was knighted at the age of 30 for the scientific work he had already achieved.

Among Hamilton's most important contributions is the establishment of an analogy between the optical theory of systems of rays and the dynamics of moving bodies. With the further development by *Carl Gustav Jacobi* (1804–1851), this theory is generally known as the Hamilton–Jacobi Principle. By this construction, for example, it was possible to determine the 10 planetary orbits around the sun, a feat normally required the solution of 30 ordinary differential equations, by merely two equations involving Hamilton's characteristic functions. However, this method was more elegant than practical; hence for almost a century, Hamilton's great method was more praised than used [129].

This situation, however, changed when *Irwin Schrödinger* (1887–1961) introduced the revolutionary wave-function model for quantum mechanics in 1926. Schrödinger had expressed Hamilton's significance quite unequivocally: 'The modern development of physics is constantly enhancing Hamilton's name. His famous analogy between optics and mechanics virtually anticipated wave mechanics, which did not have much to add to his ideas and only had to take them more seriously … If you wish to apply modern theory to any particular problem, you must start with putting the problem in Hamiltonian form' [15].

3. Existence and uniqueness

The potential problems we solve are normally posed as boundary value problems For example, given a closed region Ω with the boundary Γ and the boundary condition

$$\phi = f(\mathbf{x}); \quad \mathbf{x} \in \Gamma \tag{13}$$

where $f(\mathbf{x})$ is a continuous function, we are asked to find a harmonic function (meaning a function satisfying the Laplace equation) $\phi(\mathbf{x})$ that fulfills the boundary condition (13). This is known as the Dirichlet problem, named after Dirichlet. The corresponding problem of finding a harmonic function with the normal derivative boundary condition

$$\frac{\partial \phi}{\partial n} = g(\mathbf{x}); \quad \mathbf{x} \in \Gamma \tag{14}$$

where n is the outward normal of Γ, is called the Neumann problem, after *Carl Gottfried Neumann* (1832–1925).

The question of whether a solution of a Dirichlet or a Neumann problem exists, and when it exists, whether it is unique or not, is of great importance in mathematics and physics alike. Obviously, if we cannot guarantee the existence of a solution, the effort of finding it can be in vain. If a solution exists, but may not be unique, then we cannot tie the solution to the unique physical state that we are modeling.

The question of uniqueness is easier to answer: for the Dirichlet problem, if a solution exists, it is unique; and for the Neumann problem, it is unique to within an arbitrary constant. The existence theorem, however, is more difficult to prove (see the classical monograph *Foundations of Potential Theory* [100] by *Oliver Dimon Kellogg* (1878–1932) for the full exposition.)

To a physicist, the existence question seems to be moot. We may argue that if the mathematical problem correctly describes a physical problem, then a mathematical solution must exist, because the physical state exists. For example, Green in his 1828 seminal work [81], in which he developed Green's identities and Green's functions, presented a similar argument. He reasoned that if for a given closed region Ω, there *exists* a harmonic function U (an assumption that will be justified later) that satisfies the boundary condition

$$U = -\frac{1}{r} \quad \text{on} \quad \Gamma \tag{15}$$

then one can define the function

$$G = \frac{1}{r} + U \tag{16}$$

It is clear that G satisfy the the Laplace equation everywhere except at the pole, where it is singular. Furthermore, G takes the null value at the boundary Γ. G is known as the Green's function. Green went on to prove that for a harmonic function ϕ, whose boundary value is given by a continuous function $\phi(\mathbf{x})$, $\mathbf{x} \in \Gamma$, its solution is represented by the boundary integral equation [100,166]

$$\phi(\mathbf{x}) = -\frac{1}{4\pi} \iint_{\Gamma} \phi \frac{\partial G}{\partial n} \mathrm{d}S; \quad \mathbf{x} \in \Omega \tag{17}$$

where dS denotes surface integral. Since (17) gives the solution of the Dirichlet boundary value problem, hence the solution exists!

The above proof hinges on the existence of U, which is taken for granted at this point. How can we be sure that U exists for an arbitrary closed region Ω? Green argued that U is nothing but the electrical potential created by the charge on a grounded sheet conductor, whose shape takes the form of Γ, induced by a single charge located inside Ω. This physical state obviously exists; hence U must exist! It seems that the Dirichlet problem is proven. But is it?

In fact, mathematicians can construct counter examples for which a solution does not exist. An example was presented by *Henri Léon Lebesgue* (1875–1941)[112], which can be described as follows. Consider a deformable body whose surface is pushed inward by a sharp spine. If the tip of the deformed surface is sharp enough, for example, given by the revolution of the curve $y = \exp(-1/x)$ (see Fig. 2 for a two-dimensional projection), then the tip is an exceptional point and the Dirichlet problem is not always solvable. (See Kellogg [100] for more discussion). Furthermore, if the deformed surface closes onto itself to become a single line protruding into the body, then a Dirichlet condition cannot be arbitrarily prescribed on this degenerated boundary, as it is equivalent to prescribing a value inside the domain!

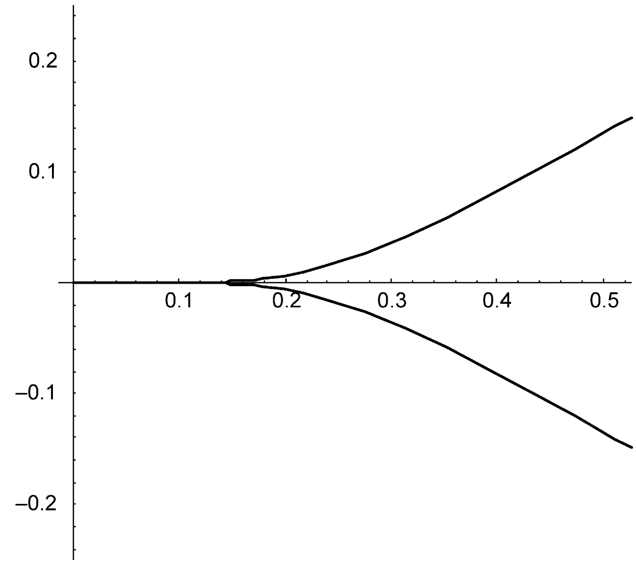


Fig. 2. Side view of a surface deformed by a sharp spine. The curve is given by $y = \exp(-1/x)$.

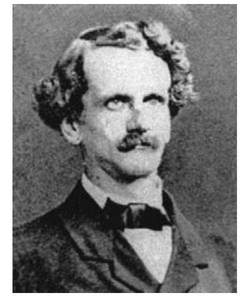
Generally speaking, the existence and uniqueness theorem for potential problems has been proven for interior and exterior boundary value problems of the Dirichlet, Neumann, Robin, and mixed type, if the bounding surface and the boundary condition satisfy certain smoothness condition [97,100]. (For interior Neumann problem, the uniqueness is only up to an arbitrarily additive constant.) For the existing proofs, the bounding surface Γ needs to be a 'Liapunov surface', which is a surface in the $C^{1,\alpha}$ continuity class, where $0 \le \alpha < 1$. Put it simply, the smoothness of the surface is such that on every point there exists a tangent plane and a normal, but not necessarily a curvature. Corners and edges, on which a tangent plane does not exist, are not allowed in this class. This puts great restrictions on the type of problems that one can solve. On the other hand, in numerical solutions such as the finite element method and the boundary element method, the solution is often sought in the weak sense by minimizing an energy norm in some sense, such as the well-known Galerkin scheme. In this case, the existence theorem has been proven for surface Γ in the $C^{0,1}$ class [42], known as the Lipschitz surface, which is a more general class than the Liapunov surface, such that edges and corners are allowed in the geometry.

3.1. Dirichlet

Johann Peter Gustav Lejeune Dirichlet (1805–1859) was born in Düren, French Empire (present day Germany). He attended the Jesuit College in Cologne at the age of 14. There he had the good fortune to be taught by *Georg Simon Ohm* (1789–1854). At the age of 16 Dirichlet entered the Collège de France in Paris, where he had the leading mathematicians at that time as teachers. In 1825, he published his first paper proving a case in Fermat's Last Theorem, which gained him instant fame. Encouraged by *Alexander von Humboldt* (1769–1859), who made recommendations on his behalf, Dirichlet returned to Germany the same year seeking a teaching position. From 1827 Dirichlet taught at the University of Breslau. Again with von Humboldt's help, he moved to Berlin in 1828 where he was appointed in the Military College. Soon afterward, he was appointed a Professor at the University of Berlin where he remained from 1828 to 1855. Dirichlet was elected to the Berlin Academy of Sciences in 1831. An improved salary from the university put him in a position to marry, and he married Rebecca Mendelssohn, one of the composer Felix Mendelssohn's sisters. Dirichlet had a lifelong friendship with Jacobi, who taught at Königsberg, and the two exerted considerable influence on each other in their researches in number theory. Dirichlet had a high teaching load and in 1853 he complained in a letter to his pupil *Leopold Kronecker* (1823–1891) that he had 13 lectures a week to give, in addition to many other duties. It was therefore a relief when, on Gauss's death in 1855, he was offered his chair at Göttingen. Sadly he was not to enjoy this new position for long. He died in 1859 after a heart attack.

Dirichlet made great contributions to the number theory. The analytic number theory may be said to begin with him. In mechanics he investigated Laplace's problem on the stability of the solar system, which led him to the Dirichlet problem concerning harmonic functions with given boundary conditions. Dirichlet is also well known for his papers on conditions for the convergence of trigonometric series. Because of this work Dirichlet is considered the founder of the theory of Fourier series [128].

3.2. Neumann



Carl Gottfried Neumann (1832–1925) was the son of *Franz Neumann* (1798–1895), a famous physicist who made contributions in thermodynamics. His mother was a sister-in-law of *Friedrich Wilhelm Bessel* (1784–1846). Neumann was born in Königsberg where his father was the Professor of Physics at the university. Neumann entered the University of Königsberg and received his doctorate in 1855. He worked on his habilitation at the University of Halle in 1858. He taught several universities, including Halle, Basel, and Tübingen. Finally, he moved to a chair at the University of Leipzig in 1868, and would stay there until his retirement in 1911.

He worked on a wide range of topics in applied mathematics such as mathematical physics, potential theory, and electrodynamics. He also made important pure mathematical contributions such as the order of connectivity of Riemann surfaces. During the 1860s Neumann wrote papers on the Dirichlet principle, in which he coined the term 'logarithmic potential' [128].

3.3. Kellogg

Oliver Dimon Kellogg (1878–1932) was born at Linnwood, Pennsylvania. His interest in mathematics was aroused as an undergraduate at Princeton University, where he received his BA in 1899. He was awarded a fellowship for graduate studies and obtained a Master degree in 1900 at Princeton. The same fellowship allowed him to spend the next year at the University of Berlin. He then moved to Göttingen to pursue his doctorate. He attended lectures by *David Hilbert* (1862–1943). At that time, *Erik Ivar Fredholm* (1866–1927) had just made progress in proving the existence of Dirichlet problem using integral equations. Hilbert was excited about the development and suggested Kellogg to undertake research on the Dirichlet problem for boundary containing corners, where Fredholm's solution did not apply. Kellogg, however, failed to answer the question satisfactorily in his thesis and several subsequent papers. With the realization of his errors, he never referred to these papers in his later work. Kellogg was hard to blame because similar errors were later made by both Hilbert and *Jules Henri Poincaré* (1854–1912), and to this date the proof of Dirichlet problem for boundary containing corners has not been accomplished.

Kellogg received his PhD in 1903 and returned to the United States to take up a post of instructor in mathematics at Princeton. Two years later he joined the University of Missouri as an Assistant Professor. He spent the next 14 fruitful years at Missouri until he was called by Harvard University in 1919. Kellogg continued to work at Harvard until his death from a heart attack suffered while climbing [13,59]. His book 'Foundations of Potential Theory' [100], first published in 1929, remains among the most authoritative work to this date.

4. Reduction in dimension and Green's formula

A key to the success of boundary element method is the reduction of spatial dimension in its integral equation representation, leading to a more efficient numerical discretization. One of the most celebrated technique of this type is the divergence theorem, which transforms a volume integral into a surface integral

$$\iiint_{\Omega} \nabla\cdot\mathbf{A}\,\mathrm{d}V = \iint_{\Gamma} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S \tag{18}$$

where $\mathbf{A}$ is a vector, $\mathbf{n}$ is the unit outward normal of Γ, and $\mathrm{d}V$ stands for volume integral. Early development of this type was found in the work of Lagrange [110] and Laplace. Eq. (18), also called Gauss's theorem, is commonly attributed to Gauss [70]. However, Gauss in 1813 only presented a few special cases in the form [99]

$$\iint_{\Gamma} n_x\,\mathrm{d}S = 0 \tag{19}$$

where n_x is the x-component of outward normal, and

$$\iint_{\Gamma} \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S = 0 \tag{20}$$

where the components of $\mathbf{A}$ are given by $A_x = A_x(y,z)$, $A_y = A_y(x,z)$, and $A_z = A_z(x,y)$. The general theorem should be credited to *Mikhail Vasilevich Ostrogradski* (1801–1862), who in 1826 presented the following result to the Paris Académie des Sciences [99]

$$\iiint_{\Omega} \mathbf{a}\cdot\nabla\phi\,\mathrm{d}V = \iint_{\Gamma} \phi\mathbf{a}\cdot\mathbf{n}\,\mathrm{d}S \tag{21}$$

where $\mathbf{a}$ is a constant vector.

Another useful formula is the Stokes's theorem, presented by *George Gabriel Stokes* (1819–1903), which transforms a surface integral into a contour integral [162]

$$\iint_{S} (\nabla\times\mathbf{A})\cdot\mathbf{n}\,\mathrm{d}S = \int_{C} \mathbf{A}\cdot\mathrm{d}\mathbf{s} \tag{22}$$

where S is an open, two sided curve surface, C is the closed contour bounding S, and $\mathrm{d}\mathbf{s}$ denotes line integral.

The most important work related to the boundary integral equation solving potential problems came from George Green, whose groundbreaking work remained obscure during his lifetime, and he earned his fame only posthumously. Green in 1828 [81] presented the three Green's identities. The first identity is

$$\iiint_{\Omega} (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,\mathrm{d}V = \iint_{\Gamma} \phi\frac{\partial\psi}{\partial n}\mathrm{d}S \tag{23}$$

The above equation easily leads to the second identity

$$\iiint_{\Omega} (\phi\nabla^2\psi - \psi\nabla^2\phi)\,\mathrm{d}V = \iint_{\Gamma} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}S \tag{24}$$

Using the fundamental solution of Laplace equation $1/r$ in (24), the third identity is obtained

$$\phi = \frac{1}{4\pi}\iint_{\Gamma} \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial(1/r)}{\partial n}\right]\mathrm{d}S \tag{25}$$

which is exactly the formulation of the present-day boundary element method for potential problems.

4.1. Gauss

Carl Friedrich Gauss (1777–1855) was born an infant prodigy into a poor and unlettered family. According to a well-authenticated story, he corrected an error in his father's payroll calculations as a child of three. He was supported by the Duke Ferdinand of Braunschweig to receive his education. Even as a student, he made major discoveries, including the method of least squares and the discovery of how to construct the regular 17-gon. However, his early career was not very successful and had to continue to rely on the financial support of his benefactor. At the age of 22, he published as his doctoral thesis the most celebrated work, the *Fundamental Theorem of Algebra*. In 1807 Gauss was finally able to secure a position as the Director of the newly founded observatory at the Göttingen University, a job he held for the rest of his life.

Gauss devoted more of his time in theoretical astronomy than in mathematics. This is considered a great loss for mathematics—just imagine how much more mathematics he could have accomplished. He devised a procedure for calculating the orbits planetoids that included the use of least square that he developed. Using his superior method, Gauss redid in an hour's time the calculation on which Euler had spent 3 days, and which sometimes was said to have led to Euler's loss of sight. Gauss remarked unkindly, '*I should also have gone blind if I had calculated in that fashion for 3 days*'.

Gauss not only adorned every branches of pure mathematics and was called the Prince of Mathematicians, he also pursued work in several related scientific fields, notably physics, mechanics, and astronomy. Together with *Wilhelm Eduard Weber* (1804–1891), he studied electromagnetism. They were the first to have successfully transmitted telegraph [30,31].

4.2. Green

George Green (1793–1841) was virtually unknown as a mathematician during his lifetime. His most important piece of work was discovered posthumously. As the son of a semi-literate, but well-to-do Nottingham baker and miller, Green was sent to a private academy at the age of eight, and left school at nine. This was the only formal education that he received until adulthood. For the next 20 years after leaving primary school, no one knew how, and from whom Green could have acquainted himself to the advanced mathematics of his day in a backwater place like Nottingham. Even the whole country of England in those days was scientifically depressed as compared to the continental Europe. Hence it was a mystery how Green could have produced as his first publication such a masterpiece without any guidance.

The next time there existed a record about Green was in 1823. At the age of 30, he joined the Nottingham Subscription Library as a subscriber. In the library he had access to books and journals. Also he had the opportunity to meet with people from the higher society. The next 5 years was not easy for Green; he had to work full time in the mill, had two daughters born (he had seven children with Jane Smith, but never married her), and his mother died in 1825. Despite these difficulties in life and his flimsy mathematical background, in 1828 he self-published one of the most important mathematical works of all times—*An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* [81]. The essay had 51 subscribers, each paid 7 shillings 6 pence, a sum equal to a poor man's weekly wage, for a work which they could hardly understood a word. One subscriber, Sir Edward Bromhead, however, was impressed by Green's prowess in mathematics. He encouraged and recommended Green to attend Cambridge University.

Several years later, Green finally enrolled at Cambridge University at the age of 40. From 1833 to 1836, Green wrote three more papers, two on electricity published by the Cambridge Philosophical Society, and one on hydrodynamics published by the Royal Society of Edinburgh. After graduating in 1837, he stayed at Cambridge for a few years to work on his own mathematics and to wait for an appointment. In 1838–1839 he had two papers in hydrodynamics, two papers on reflection and refraction of light, and two papers on sound [82]. In 1839, he was elected to a Parse Fellowship at Cambridge, a junior position. Due to poor health, he had to return to Nottingham in 1840. He died in 1841 at the age of 47. At the time of his death, his work was virtually unknown.

At the year of Green's death, *William Thomson* (*Lord Kelvin*) (1824–1907) was admitted to Cambridge. While studying the subject of electricity as a part of preparation for his Senior Wrangler exam, he first noticed the existence of Green's paper in a footnote of a paper by Robert Murphy. He started to look for a copy, but no one knew about it. After his graduation in 1845, and before his departure to France to enrich his education, he mentioned it to his teacher *William Hopkins* (1793–1866). It happened that Hopkins had three copies. Thomson was immediately excited about what he had read in the paper. He brought the article to Paris and showed it to *Jacques Charles François Sturm* (1803–1855) and *Joseph Liouville* (1809–1882). Later Thomson republished Green's essay, rescuing it from sinking into permanent obscurity [33].

Green's 1828 essay had profoundly influenced Thomson and *James Clerk Maxwell* (1831–1879) in their study of electrodynamics and magnetism. The methodology has also been applied to many classical fields of physics such as acoustics, elasticity, and hydrodynamics. During the bicentennial celebration of Green's birth in 1993, physicists *Julian Seymour Schwinger* (1918–1994) and *Freeman J. Dyson* (1923-) delivered speeches on the role of Green's functions in the development of 20th century quantum electrodynamics [33].

4.3. Ostrogradski

Mikhail Vasilevich Ostrogradski (1801–1862) was born in Pashennaya, Ukraine. He entered the University of Kharkov in 1816 and studied physics and mathematics. In 1822 he left Russia to study in Paris. Between 1822 and 1827 he attended lectures by Laplace, Fourier, *Adrien-Marie Legendre* (1752–1833), Poisson, and *Augustin-Louis Cauchy* (1789–1857). He made rapid progress in Paris and soon began to publish papers in the Paris Academy. His papers at this time showed the influence of the mathematicians in Paris and he wrote on physics and the integral calculus. These papers were later incorporated in a major work on hydrodynamics, which he published in Paris in 1832.

Ostrogradski went to St Petersburg in 1828. He presented three important papers on the theory of heat, double integrals and potential theory to the Academy of Sciences. Largely on the strength of these papers he was elected an academician in the applied mathematics section. In 1840 he wrote on ballistics and introduced the topic to Russia. He was considered as the founder of the Russian school of theoretical mechanics [128].

4.4. Stokes

George Gabriel Stokes (1819–1903) was born in Skreen, County Sligo, Ireland. In 1837 he entered Pembroke College of Cambridge University. He was coached by William Hopkins, who had among his students Thomson, Maxwell, and *Peter Guthrie Tait* (1831–1901), and had the reputation as the 'senior wrangler maker.' In 1841 Stokes graduated as Senior Wrangler (the top First Class degree) and was also the first Smith's prizeman. Pembroke College immediately gave him a fellowship.

Inspired by the recent work of Green, Stokes started to undertake research in hydrodynamics and published papers on the motion of incompressible fluids in 1842. After completing the research Stokes discovered that *Jean Marie Constant Duhamel* (1797–1872) had already obtained similar results for the study of heat in solids. Stokes continued his investigations, looking into the internal friction in fluids in motion. After he had deduced the correct equations of motion, Stokes discovered that again he was not the first to obtain the equations, since *Claude Louis Marie Henri Navier* (1785–1836), Poisson and *Adhémar Jean Claude Barré de Saint-Venant* (1797–1886) had already considered the problem. Stokes decided that his results were sufficiently different and published the work in 1845. Today the fundamental equation of hydrodynamics is called the Navier–Stokes equations. The viscous flow in slow motion is called Stokes flow. The mathematical theorem that carries his name, Stokes theorem, first appeared in print in 1854 as an examination question for the Smith's Prize. It is not known whether any student could answer the question at that time.

In 1849 Stokes was appointed the Lucasian Professor of Mathematics at Cambridge, the chair Newton once held. In 1851 Stokes was elected to the Royal Society, and was awarded the Rumford Medal in 1852. He was appointed Secretary of the Society in 1854, which he held until 1885. He was the President of the Society from 1885 to 1890. Stokes received the Copley Medal from the Royal Society in 1893, and served as the Master of Pembroke College in 1902–1903 [128].

5. Integral equations

Inspired by the use of influence functions as a method for solving problems of beam deflection subject to distributed load, Fredholm started the investigation of integral equations 73. Fredholm [67] proved in 1903 the existence and uniqueness of solution of the linear integral equation

$$\mu(x) - \lambda \int_a^b K(x,\xi)\mu(\xi)\mathrm{d}\xi = f(x); \quad a \le x \le b \tag{26}$$

where λ is a constant, $f(x)$ and $K(x,\xi)$ are given continuous functions, and $\mu(x)$ is the solution sought. Eq. (26) is known as the Fredholm integral equation of the second kind.

By the virtue of the above Fredholm theorem, we can solve the Dirichlet problems by the following formula [161]

$$\phi(\mathbf{x}) = \mp 2\pi\mu(\mathbf{x}) + \iint_\Gamma K(\mathbf{x},\xi)\mu(\xi)\mathrm{d}S(\xi); \quad \mathbf{x}\in\Gamma \tag{27}$$

In the above the upper sign corresponds to the interior problem, the lower sign the exterior problem, μ is the distribution density, Γ is a closed Liapunov surface, $\phi(\mathbf{x})$ is the Dirichlet boundary condition, and the kernel K is given by

$$K(\mathbf{x},\xi) = \frac{\partial}{\partial n(\xi)}\left[\frac{1}{r(\mathbf{x},\xi)}\right] \tag{28}$$

The kernel is known as a dipole, or a 'double-layer potential'. The Fredholm theorem guarantees the existence and uniqueness of μ. Once the distribution density μ is solved from (27) by some technique, the full solution of the boundary value problem is given by

$$\phi(\mathbf{x}) = \iint_\Gamma \frac{\partial[1/r(\mathbf{x},\xi)]}{\partial n(\xi)}\mu(\xi)\mathrm{d}S(\xi); \quad \mathbf{x}\in\Omega \tag{29}$$

which is a continuous distribution of the double-layer potential on the boundary.

For the Neumann problem, we can utilize the following boundary equation:

$$\frac{\partial\phi(\mathbf{x})}{\partial n(\mathbf{x})} = \pm 2\pi\sigma(\mathbf{x}) + \iint_\Gamma K(\xi,\mathbf{x})\sigma(\xi)\mathrm{d}S(\xi); \quad \mathbf{x}\in\Gamma \tag{30}$$

Here again the upper and lower sign, respectively, corresponds to the interior and exterior problems, σ is the distribution density, Γ is the bounding Liapunov surface, $\partial\phi/\partial n$ is the Neumann boundary condition, and the kernel is given by

$$K(\xi,\mathbf{x}) = \frac{\partial}{\partial n(\mathbf{x})}\left[\frac{1}{r(\mathbf{x},\xi)}\right] \tag{31}$$

After solving for σ, the potential for the whole domain is given by

$$\phi(\mathbf{x}) = \iint_{\Gamma} \frac{1}{r(\mathbf{x}, \boldsymbol{\xi})} \sigma(\boldsymbol{\xi}) \mathrm{d}S(\boldsymbol{\xi}); \quad \mathbf{x} \in \Omega \tag{32}$$

which is the distribution of the source, or the 'single-layer potential', on the boundary. Fredholm suggested a discretization procedure to solve the above equations. However, without a fast enough computer to solve the resultant matrix system, the idea was impractical; hence further development of utilizing these equations was limited to analytical work.

For mixed boundary value problems, a pair of integral equations is needed. For the 'single-layer method' applied to interior problems, the following pair

$$\phi(\mathbf{x}) = \iint_{\Gamma} \frac{1}{r(\mathbf{x}, \xi)} \sigma(\xi) \mathrm{d}S(\xi); \quad \mathbf{x} \in \Gamma_{\phi} \tag{33}$$

$$\frac{\partial \phi(\mathbf{x})}{\partial n(\mathbf{x})} = \iint_{\mathrm{CPV}} \frac{\partial [1/r(\mathbf{x}, \boldsymbol{\xi})]}{\partial n(\mathbf{x})} \sigma(\boldsymbol{\xi}) \mathrm{d}S(\boldsymbol{\xi}); \quad \mathbf{x} \in \Gamma_q \tag{34}$$

can be, respectively, applied on the Dirichlet part Γ_{ϕ} and the Neumann part Γ_q of the boundary. We notice that (33) contains a weak (integrable) singularity as $\boldsymbol{\xi} \to \mathbf{x}$; while (34) contains a strong (non-integrable) singularity. The integral in (34) needs to be interpreted in the 'Cauchy principal value' sense, which is denoted as CPV under the integral sign. On a smooth part of the boundary not containing edges and corners, the result of the Cauchy principle value limit is just (30). This idea of interpreting and handling this type of strong singularity was introduced by Cauchy in 1814 [34].

A 'double-layer method' can also be formulated to solve mixed boundary value problems using the following pair of equations

$$\phi(\mathbf{x}) = \iint_{\mathrm{CPV}} \frac{\partial [1/r(\mathbf{x}, \boldsymbol{\xi})]}{\partial n(\mathbf{x})} \mu(\boldsymbol{\xi}) \mathrm{d}S(\boldsymbol{\xi}); \quad \mathbf{x} \in \Gamma \tag{35}$$

$$\frac{\partial \phi(\mathbf{x})}{\partial n(\mathbf{x})} = \iint_{\mathrm{HFP}} \frac{\partial}{\partial n(\mathbf{x})} \left[\frac{\partial 1/r(\mathbf{x}, \boldsymbol{\xi})}{\partial n(\boldsymbol{\xi})} \right] \mu(\boldsymbol{\xi}) \mathrm{d}S(\boldsymbol{\xi}); \quad \mathbf{x} \in \Gamma \tag{36}$$

The integral in (36) contains a 'hypersingularity' and is marked with HFP under the integral sign, standing for 'Hadamard finite part.' This concept was introduced by *Jacques Salomon Hadamard* (1865–1963) in 1908 [85].

In boundary element terminology, the single- and double-layer methods are referred to as the 'indirect methods,' because the distribution density μ or σ, not the potential ϕ itself, is solved. The numerical method based on Green's third identity (25), which solves ϕ or $\partial\phi/\partial n$ on the boundary, is called the 'direct method'.

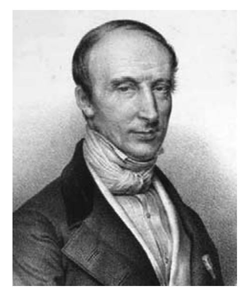
It is of interest to mention that for a Dirichlet problem, the single-layer method reduces to using (33) only. Eq. (33) however is a Fredholm integral equation of the first kind, whose solution is unstable [175]. In that case, the second kind equation, (27) or (35), should be used.

Similar integral representation exists in the complex variable domain. Cauchy in 1825 [35] presented one of the most important theorems in complex variable—the Cauchy integral theorem, from which came the Cauchy integral formula, expressed as

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta \tag{37}$$

where z and ζ are complex variables, f is an analytic function, and C is a smooth, closed contour in the complex plane. When z is located on the contour, $z \in C$, Eq. (37) can be exploited for the numerical solution of boundary value problems, a procedure known as the complex variable boundary element method.

5.1. Cauchy



Augustin-Louis Cauchy (1789–1857) was born in Paris during the difficult time of French Revolution. Cauchy's father was active in the education of young Augustin-Louis. Laplace and Lagrange were frequent visitors at the Cauchy family home, and Lagrange particularly took interest in Cauchy's mathematical ability. In 1805 Cauchy took the entrance examination of the Ecole Polytechnique and was placed second. In 1807 he entered Ecole des Ponts et Chaussées to study engineering, specializing in highways and bridges, and finished school in 2 years. At the age of 20, he was appointed as a Junior Engineer to work on the construction of Port Napoléon in Cherbourg. He worked there for 3 years and performed excellently. In 1812, he became ill and decided to returned to Paris to seek a teaching position.

Cauchy's initial attempts in seeking academic appointment were unsuccessful. Although he continued to publish important pieces of mathematical work, he lost to Legendre, to *Louis Poinsot* (1777–1859), and to *André Marie Ampère* (1775–1836) in competition for academic positions. In 1814 he published the memoir on definite integrals that later became the basis of his theory of complex functions. In 1815 Cauchy lost out to *Jacques Philippe Marie Binet* (1786–1856) for a mechanics chair at the Ecole Polytechnique, but then he was finally appointed Assistant Professor of Analysis there. In 1816 he won the Grand Prix of the Académie des Sciences for a work on waves, and was later admitted to the Académie. In 1817, he was able to substitute for *Jean-Baptiste Biot* (1774–1862), Chair of Mathematical Physics at the Collège de France, and later for Poisson. It was not until 1821 that he was able to obtain a full position replacing Ampère.

Cauchy was staunchly Catholic and was politically a royalist. By 1830 the political events in Paris forced him to leave Paris for Switzerland. He soon lost all his positions in Paris. In 1831 Cauchy went to Turin and later accepted an offer to become a Chair of Theoretical Physics. In 1833 Cauchy went from Turin to Prague, and returned to Paris in 1838. He regained his position at the Académie but not his teaching positions because he had refused to take an oath of allegiance to the new regime. Due to his political and religious views, he continued to have difficulty in getting appointment.

Cauchy was probably next to Euler the most published author in mathematics, having produced five textbooks and over 800 articles. Cauchy and his contemporary Gauss were credited for introducing rigor into modern mathematics. It was said that when Cauchy read to the Académie des Sciences in Paris his first paper on the convergence of series, Laplace hurried home to verify that he had not made mistake of using any divergence series in his *Mécanique Céleste*. The formulation of elementary calculus in modern textbooks is essentially what Cauchy expounded in his three great treatises: *Cours d'Analyse de l'École Royale Polytechnique* (1821), *Résumé des Leçons sur le Calcul Infinitésimal* (1823), and *Leçons sur le Calcul Différentiel* (1829). Cauchy was also credited for setting the mathematical foundation for complex variable and elasticity. The basic equation of elasticity is called the Navier–Cauchy equation [8,79].

5.2. Hadamard

Jacques Salomon Hadamard (1865–1963) began his schooling at the Lycée Charlemagne in Paris, where his father taught. In his first few years at school he was not good at mathematics; he wrote in 1936: '… *in arithmetic, until the fifth grade, I was last or nearly last*'. It was a good mathematics teacher turned him around. In 1884 Hadamard was placed first in the entrance examination for École Normale Supérieure, where he obtained his doctorate in 1892. His thesis on functions of a complex variable was one of the first to examine the general theory of analytic functions, in particular it contained the first general work on singularities. In the same year Hadamard received the Grand Prix des Sciences Mathématique for his paper 'Determination of the number of primes less than a given number'. The topic proposed for the prize, concerning filling gaps in work of *Bernhard Riemann* (1826–1866) on zeta functions, had been put forward by *Charles Hermite* (1822–1901) with his friend *Thomas Jan Stieltjes* (1856–1894) in mind to win it. However, Stieltjes discovered a gap in his proof and never submitted an entry. The next 4 years Hadamard was first a lecturer at Bordeaux, and then promoted to Professor of Astronomy and Rational Mechanics in 1896. During this time he published his famous determinant inequality; matrices satisfying this relation are today called Hadamard matrices, which are important in the theory of integral equations, coding theory, and other areas.

In 1897 Hadamard resigned his chair in Bordeaux and moved to Paris to take up posts in Sorbonne and Collège de France. His research turned more toward mathematical physics; yet he always argued strongly that he was a mathematician, not a physicist. His famous 1898 work on geodesics on surfaces of negative curvature laid the foundations of symbolic dynamics. Among the other topics he considered were elasticity, geometrical optics, hydrodynamics and boundary value problems. He introduced the concept of a well-posed initial value and boundary value problem. Hadamard continued to receive prizes for his research and was honored in 1906 with the election as the President of the French Mathematical Society. In 1909 he was appointed to the Chair of Mechanics at the Collège de France. In the following year he published *Leçons sur le calcul des variations*, which helped lay the foundations of functional analysis (the word 'functional' was introduced by him). Then in 1912 he was appointed as Professor of Analysis at the École Polytechnique. Near the end of 1912 Hadamard was elected to the Academy of Sciences to succeed Poincaré. After the start of World War II, when France fell to Germany in 1940, Hadamard, being a Jew, escaped to the United States where he was appointed to a visiting position at Columbia University. He left America in 1944 and spent a year in England before returning to Paris after the end of the war. He was lauded as one of the last universal mathematicians whose contributions broadly span the fields of mathematics. He lived to 98 year old [118,128].

5.3. Fredholm

Erik Ivar Fredholm (1866–1927) was born in Stockholm, Sweden. After his baccalaureate, Fredholm enrolled in 1886 at the University of Uppsala, which was the only doctorate granting university in Sweden at that time. Through an arrangement he studied under *Magnus Gösta Mittag-Leffler* (1846–1927) at the newly founded University of Stockholm, and acquired his PhD from the University of Uppsala in 1893. Fredholm's first publication 'On a special class of functions' came in 1890. It so impressed Mittag-Leffler that he sent a copy of the paper to Poincaré. In 1898 he received the degree of Doctor of Science from the same university.

Fredholm is best remembered for his work on integral equations and spectral theory. Although *Vito Volterra* (1860–1940) before him had studied the integral equation theory, it was Fredholm who provided a more thorough treatment. This work was accomplished during the months of 1899 which Fredholm spent in Paris studying the Dirichlet problem with Poincaré, *Charles Emile Picard* (1856–1941), and Hadamard. In 1900 a preliminary report was published and the work was completed in 1903 [67]. Fredholm's contributions quickly became well known. Hilbert immediately saw the importance and extended Fredholm's work to include a complete eigenvalue theory for the Fredholm integral equation. This work led directly to the theory of Hilbert spaces.

After receiving his Doctor of Science degree, Fredholm was appointed as a Lecturer in mathematical physics at the University of Stockholm. He spent his whole career at the University of Stockholm being appointed to a chair in mechanics and mathematical physics in 1906. In 1909–1910 he was Pro-Dean and then Dean in Stockholm University.

Fredholm wrote papers with great care and attention so he produced work of high quality that quickly gained him a high reputation throughout Europe. However, his papers required so much effort that he wrote only a few. In fact, his *Complete Works* in mathematics comprises of only 160 pages. After 1910 he wrote little beyond revisiting his earlier work [128].

6. Extended Green's formula

Green's formula (25), originally designed to solve electrostatic problems, was such a success that the idea was followed to solve many other physical problems [166]. For example, *Hermann Ludwig Ferdinand von Helmholtz* (1821–1894) in his study of acoustic problems presented the following equation in 1860 [87], known as the Helmholtz equation

$$\nabla^2\phi + k^2\phi = 0 \tag{38}$$

where k is a constant known as the wave number. He also derived the fundamental solution of (38) as

$$\phi = \frac{\cos kr}{r} \tag{39}$$

In the same paper he established the equivalent Green's formula

$$\phi = \frac{1}{4\pi}\iint\limits_{\Gamma}\left[\frac{\cos kr}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{\cos kr}{r}\right)\right]\mathrm{d}S \tag{40}$$

which can be compared to (25).

For elasticity, an important step toward deriving Green's formula was made by *Enrico Betti* (1823–1892) in 1872, when he introduced the reciprocity theorem, one of the most celebrated relation in mechanics [10]. The theory can be stated as follows: given two independent elastic states in a static equilibrium, $(\mathbf{u}, \mathbf{t}, \mathbf{F})$ and $(\mathbf{u}', \mathbf{t}', \mathbf{F}')$, where $\mathbf{u}$ and $\mathbf{u}'$ are displacement vectors, $\mathbf{t}$ and $\mathbf{t}'$ are tractions on a closed surface Γ, and $\mathbf{F}$ and $\mathbf{F}'$ are body forces in the enclosed region Ω, they satisfy the following reciprocal relation

$$\iint\limits_{\Gamma}(\mathbf{t}'\cdot\mathbf{u} - \mathbf{t}\cdot\mathbf{u}')\mathrm{d}S = \iiint\limits_{\Omega}(\mathbf{F}\cdot\mathbf{u}' - \mathbf{F}'\cdot\mathbf{u})\mathrm{d}V \tag{41}$$

The above theorem, known as the Betti–Maxwell reciprocity theorem, was a generalization of the reciprocal principle derived earlier by Maxwell [117] applied to trusses. *John William Strutt* (*Lord Rayleigh*) (1842–1919) further generalized the above theorem to elastodynamics in the frequency domain, and also extended the forces and displacements concept to generalized forces and generalized displacements [136,137].

In the same sequence of papers [10,11], Betti presented the fundamental solution known as the center of dilatation [114]

$$\mathbf{u}^* = \frac{1-2\nu}{8\pi G(1-\nu)}\nabla\left(\frac{1}{r}\right) \tag{42}$$

where G is the shear modulus, and ν is the Poisson ratio. The use of (42) in (41) produced the integral representation for dilatation

$$e = \nabla\cdot\mathbf{u} = \iint\limits_{\Gamma}(\mathbf{t}\cdot\mathbf{u}^* - \mathbf{t}^*\cdot\mathbf{u})\mathrm{d}S + \iiint\limits_{\Omega}\mathbf{F}\cdot\mathbf{u}^*\,\mathrm{d}V \tag{43}$$

where $\mathbf{t}^*$ is the boundary traction vector of the fundamental solution (42).

The more useful formula that gives the integral equation representation of displacements, rather than dilatation, requires the fundamental solution of a point force in infinite space, which was provided by Kelvin in 1848 [101]

$$u_{ij}^* = \frac{1}{16\pi G(1-\nu)}\frac{1}{r}\left[\frac{x_i x_j}{r^2} + (3-4\nu)\delta_{ij}\right] \tag{44}$$

where δ_{ij} is the Kronecker delta. In the above we have switched to the tensor notation, and the second index in u_{ij}^* indicates the direction of the applied point force. Utilizing (44), *Carlo Somigliana* (1860–1955) in 1885 [157] developed the following integral representation for displacements

$$u_j = \iint\limits_{\Gamma}(t_i u_{ij}^* - t_{ij}^* u_i)\mathrm{d}S + \iiint\limits_{\Omega}F_i u_{ij}^*\,\mathrm{d}V \tag{45}$$

Eq. (45), called the Somigliana identity, is the elasticity counterpart of Green's formula (25).

Volterra [183] in 1907 presented the dislocation solution of elasticity, as well as other singular solutions such as the force double and the disclination, generally known as the nuclei of strain [114]. Further dislocation solutions were given by Somigliana in 1914 [158] and 1915 [159]. For a point dislocation in unbounded three-dimensional space, the resultant displacement field is

$$u^*_{ijk} = \frac{1}{4\pi(1-\nu)} \times \frac{1}{r^2}\left[(1-2\nu)(\delta_{kj}x_i - \delta_{ij}x_k - \delta_{ik}x_j) - \frac{2}{r^2}x_i x_j x_k\right] \tag{46}$$

This singular solution can be distributed over the boundary Γ to give the Volterra integral equation of the first kind [182]

$$u_k = \iint_\Gamma u^*_{kji} n_j \mu_i \, \mathrm{d}S + \iiint_\Omega u^*_{ki} F_i \, \mathrm{d}V \tag{47}$$

where μ_i is the component of the distribution density vector $\boldsymbol{\mu}$, also known as the displacement discontinuity. Eq. (47) is equivalent to (35) of the potential problem, and can be called the double-layer method. The counterpart of the single-layer method (33) is given by the Somigliana integral equation

$$u_j = \iint_\Gamma u^*_{ji}\sigma_i \, \mathrm{d}S + \iiint_\Omega u^*_{ji} F_i \, \mathrm{d}V \tag{48}$$

where σ_i is the component of the distribution density vector $\boldsymbol{\sigma}$, known as the stress discontinuity.

Similar to Cauchy integral (37) for potential problems, the complex variable potentials and integral equation representation for elasticity exist, which was formulated by *Gury Vasilievich Kolosov* (1867–1936) in 1909 [102]. These were further developed by *Nikolai Ivanovich Muskhelishvili* (1891–1976) [125,126].

We can derive the above extended Green's formulae in a unified fashion. Consider the generalized Green's theorem [123]

$$\int_\Omega (\mathbf{v}\mathfrak{L}\{\mathbf{u}\} - \mathbf{u}\mathfrak{L}^*\{\mathbf{v}\})\mathrm{d}\mathbf{x} = \int_\Gamma (\mathbf{v}\mathfrak{B}\{\mathbf{u}\} - \mathbf{u}\mathfrak{B}^*\{\mathbf{v}\})\mathrm{d}\mathbf{x} \tag{49}$$

In the above $\mathbf{u}$ and $\mathbf{v}$ are two independent vector functions, $\mathfrak{L}$ is a linear partial differential operator, $\mathfrak{L}^*$ is its adjoint operator, $\mathfrak{B}$ is the generalized boundary normal derivative, and $\mathfrak{B}^*$ is its adjoint operator. The right hand side of (49) is the consequence of integration by parts of the left hand side operators. Eq. (49) may be compared with the Green's second identify (24). If we assume that $\mathbf{u}$ is the solution of the homogeneous equations

$$\mathfrak{L}(\mathbf{u}) = 0 \quad \text{in } \Omega \tag{50}$$

subject to certain boundary conditions, and $\mathbf{v}$ is replaced by the fundamental solution of the adjoint operator satisfying

$$\mathfrak{L}^*\{\mathbf{G}\} = \delta \tag{51}$$

Eq. (49) becomes the boundary integral equation

$$\mathbf{u} = \int_\Gamma (\mathbf{u}\mathfrak{B}^*\{\mathbf{G}\} - G\mathfrak{B}\{\mathbf{u}\})\mathrm{d}\mathbf{x} \tag{52}$$

As an example, we consider the general second order linear partial differential equation is two-dimension

$$\mathfrak{L}\{u\} = A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu \tag{53}$$

where the coefficients A,B,..., and F are functions of x and y. The generalized Green's second identity in the form of (49) exists with the definition of the operators [83]

$$\mathfrak{L}^*\{v\} = \frac{\partial^2 Av}{\partial x^2} + 2\frac{\partial^2 Bv}{\partial x \partial y} + \frac{\partial^2 Cv}{\partial y^2} - \frac{\partial Dv}{\partial x} - \frac{\partial Ev}{\partial y} + Fv \tag{54}$$

$$\mathfrak{B}\{u\} = \left(A\frac{\partial u}{\partial x} + 2B\frac{\partial u}{\partial y}\right)n_x + \left(C\frac{\partial u}{\partial y} + Eu\right)n_y \tag{55}$$

$$\mathfrak{B}^*\{v\} = \left(\frac{\partial Av}{\partial x} - Dv\right)n_x + \left(2\frac{\partial Bv}{\partial x} + \frac{\partial Cv}{\partial y}\right)n_y \tag{56}$$

If we require u and v to satisfy (50) and (51), respectively, we then obtain the boundary integral equation formulation (52).

6.1. Helmholtz

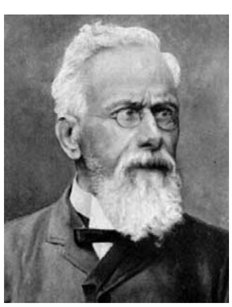
Hermann Ludwig Ferdinand von Helmholtz (1821–1894) was born in Potsdam, Germany. He attended Potsdam Gymnasium where his father was a teacher. His interests at school were mainly in physics. However, due to the financial situation of his family, he accepted a government grant to study medicine at the Royal Friedrich-Wilhelm Institute of Medicine and Surgery in Berlin. His research career began in 1841 when he worked on the connection between nerve fibers and nerve cells for his dissertation. He rejected the dominant physiology theory at that time, which was based on vital forces, and strongly argued on the ground of physics and chemistry principles. He graduated from the Medical

Institute in 1843 and had to serve as a military doctor for 10 years. He spent all his spare time doing research.

In 1847 he published the important paper '*Uber die Erhaltung der Kraft*' that established the law of conservation of energy. In the following year, Helmholtz was released from his obligation as an army doctor and became an Assistant Professor and Director of the Physiological Institute at Königsberg. In 1855, he was appointed to the Chair of Anatomy and Physiology in Bonn. Although at this time Helmholtz had gained a world reputation, complaints were made to the Ministry of Education from traditionalist that his lectures on anatomy were incompetent. Helmholtz reacted strongly to these criticisms and moved to Heidelberg in 1857 to set up a new Physiology Institute. Some of his most important work was carried out during this time.

In 1858 Helmholtz published his important paper on the motion of a perfect fluid by decomposing it into translation, rotation and deformation. His study on vortex tube played an important role in the later study of turbulence in hydrodynamics, and knot theory in topology. Helmholtz also studied mathematical physics and acoustics, producing in 1863 '*On the Sensation of Tone as a Physiological Basis for the Theory of Music*' [88]. From around 1866 Helmholtz began to move away from physiology and toward physics. When the Chair of Physics in Berlin became vacant in 1871, he was able to negotiate a new Physics Institute under his control. In 1883, he was ennobled by William I. In 1888, he was appointed as the first President of the Physikalisch-Technische Reichsanstalt at Berlin, a post that he held until his death in 1894 [32,128,192].

6.2. Betti



Enrico Betti (1823–1892) studied mathematics and physics at the University of Pisa. He graduated in 1846 and was appointed as an assistant at the university. In 1849 Betti returned to his home town of Pistoia where he became a teacher of mathematics at a secondary school. In 1854 he moved to Florence where again he taught in a secondary school. He was appointed as Professor of Higher Algebra at the University of Pisa in 1857. In the following year Betti visited the mathematical centres of Europe, including Göttingen, Berlin, and Paris, making many important mathematical contacts. In particular, in Göttingen Betti met and became friendly with Riemann. Back in Pisa he moved in 1859 to the Chair of Analysis and Higher Geometry.

During those days the political and military events in Italy were intensifying as the country came nearer to unification. In 1859 there was a war with Austria and by 1861 the Kingdom of Italy was formally created. Betti served the government of the new country as a member of Parliament. In 1863 Riemann left his post as Professor of Mathematics at Göttingen and move to Pisa, hoping that warmer weather would cure his tuberculosis. Influenced by his friend Riemann, Betti started to work on potential theory and elasticity. His famous theory of reciprocity in elasticity was published in 1872.

Over quite a number of years Betti mixed political service with service for his university. He served a term as Rector of the University of Pisa and in 1846 became the Director of its teacher's college, the Scuola Normale Superiore, a position he held until his death. Under his leadership the Scuola Normale Superiore in Pisa became the leading Italian centre for mathematical research and education. He served as an Undersecretary of State for education for a few months, and a Senator in the Italian Parliament in 1884 [128].

6.3. Kelvin

William Thomson (*Lord Kelvin*) (1824–1907) was well prepared by his father, James Thomson, Professor of Mathematics at the University of Glasgow, for his career. He attended Glasgow University at the age of 10, and later entered Cambridge University at 17. It was expected that he would won the senior wrangler position at graduation; but to his and his father's disappointment, he finished the second wrangler in 1845. The fierce competition of the 'tripos', an honors examination instituted at Cambridge in 1824, attracted many best young minds to Cambridge in those days. Among Thomson's contemporaries were Stokes, a senior wrangler in 1841, and Maxwell, a second wrangler in 1854. In 1846 the Chair of Natural Philosophy in Glasgow became vacant. Thomson's father ran a successful campaign to get his son elected to the chair at the age of 22.

Thomson was foremost among the small group of British scientists who helped to lay the foundations of modern physics. His contributions to science included a major role in the development of the second law of thermodynamics, the absolute temperature scale (measured in 'kelvins'), the dynamical theory of heat, the mathematical analysis of electricity and magnetism, including the basic ideas for the electromagnetic theory of light, the geophysical determination of the age of the Earth, and fundamental work in hydrodynamics. His theoretical work on submarine telegraphy and his inventions of mirror-galvanometer for use on submarine cables aided Britain in laying the transatlantic cable, thus gaining the lead in world communication. His participation in the telegraph cable project earned him the knighthood in 1866, and a large personal fortune [151,167].

6.4. Rayleigh

John William Strutt (*Lord Rayleigh*) (1842–1919) was the eldest son of the second Baron Rayleigh. After studying in a private school without showing extraordinary signs of scientific capability, he entered Trinity College, Cambridge, in 1861. As an undergraduate, he was coached by *Edward John Routh* (1831–1907), who had the reputation of being an outstanding teacher in mathematics and mechanics. Rayleigh (a title he did not inherit until he was thirty years old) was greatly influence by Routh, as well as by Stokes. He graduated in 1865 with top honors garnering not only the Senior Wrangler title, but also the first Smith's prizeman. Rayleigh was faced with a difficult decision: knowing that he would succeed to the title of the third Baron Rayleigh, taking up a scientific career was not really acceptable to the members of his family. By this time, however, Rayleigh was determined to devote his life to science so that his social obligations would not stand in his way.

In 1866, Rayleigh was elected to a fellowship of Trinity College. Around that time he read Helmholtz' book *On the Sensations of Tone* [88], and became interested in acoustics. Rayleigh was married in 1871, and had to give up his fellowship at Trinity. In 1872, Rayleigh had an attack of rheumatic fever and was advised to travel to Egypt for his health. He took his wife and several relatives sailed down the Nile during the last months of 1872, returning to England in the spring of 1873. It was during that trip that he started his work on the famous two volume treatise *The Theory of Sound* [137], eventually published in 1877.

Rayleigh's father died in 1873. He became the third Baron Rayleigh and had to devote part of his time supervising the estate. In 1879, Maxwell died, and Rayleigh was elected to the vacated post of Cavendish Professor of Experimental Physics at Cambridge. At the end of 1884, Rayleigh resigned his Cambridge professorship and settled in his estate. There in his self-funded laboratory he continued his intensive scientific work to the end of his life. Rayleigh made many important scientific contributions including the first correct light scattering theory that explained why the sky is blue, the theory of soliton, the surface wave known as Rayleigh wave, the hydrodynamic similarity theory, and the Rayleigh–Ritz method in elasticity. In 1904 Rayleigh won a Nobel Prize for his 1895 discovery of argon gas. He also served many public functions including being the President of the London Mathematical Society (1876–1878), President of the Royal Society of London (1905–1908), and Chancellor of Cambridge University (1908 until his death) [138,168].

6.5. Volterra

Vito Volterra (1860–1940) was born in Ancona, Italy, a city on the Adriatic Sea. When Volterra was 2-year old, his father died and he was raised by his uncle. Volterra began his studies at the Faculty of Natural Sciences of the University of Florence in 1878. In the following year he won a competition to become a student at the Scuola Normale Superiore di Pisa. In 1882, he graduated with a doctorate in physics at the University of Pisa. Among his teachers were Betti, who held the Chair of Rational Mechanics. Betti was impressed by his student that upon graduation he appointed Volterra his assistant. In 1883 Volterra was given a Professorship in Rational Mechanics at Pisa. Following Betti's death in 1892 he was also in charge of mathematical physics. From 1893 until 1900 he held the Chair of Rational Mechanics at the University of Torino. In 1900 he moved to the University of Rome, succeeding *Eugenio Beltrami* (1835–1900) as Professor of Mathematical Physics. Volterra's work encompassed integral equations, the theory of functions of a line (later called functionals after Hadamard), the theory of elasticity, integro-differential equations, the description of hereditary phenomena in physics, and mathematical biology. Beginning in 1912, Volterra regularly lectured at the Sorbonne in Paris.

In 1922, when the Fascists seized power in Italy, Volterra—a Senator of the Kingdom of Italy since 1905—was one of the few who spoke out against fascism, especially the proposed changes to the educational system. At that time (1923–1926) he was President of the Accademia Nazionale dei Lincei, and he was regarded as the most eminent man of science in Italy. As a direct result of his unwavering stand, especially his signing of the 'Intellectual's Declaration' against fascism in 1926 and, 5 years later, his refusal to swear the oath of allegiance to the fascist government imposed on all university professors, Volterra was dismissed from his chair at the University of Rome in 1931. In the following year he was deprived of all his memberships in the scientific academies and cultural institutes in Italy. From that time on he lectured and lived mostly abroad, in Paris, in Spain, and in other European countries. Volterra died in isolation on October 11, 1940 [28].

6.6. Somigliana

Carlo Somigliana (1860–1955) began his university study at Pavia, where he was a student of Beltrami. Later he transferred to Pisa and had Betti among his teachers, and Volterra among his contemporaries. He graduated from Scuola Normale Superiore di Pisa in 1881. In 1887 Somigliana began teaching as an assistant at the University of Pavia. In 1892, as the result of a competition, he was appointed as University Professor of Mathematical Physics. Somigliana was called to Turin in 1903 to become the Chair of Mathematical Physics. He held the post until his retirement in 1935, and moved to live in Milan. During the World War II, his apartment in Milan was destroyed. After the war he retreated to his family villa in Casanova Lanza and stayed active in research until near his death in 1955.

Somigliana was a classical physicist–mathematician faithful to the school of Betti and Beltrami. He made important contributions in elasticity. The Somigliana integral equation for elasticity is the equivalent of Green's formula for potential theory. He is also known for the Somigliana dislocations. His other contributions included seismic wave propagation and gravimetry [173].

6.7. Kolosov

Gury Vasilievich Kolosov (1867–1936) was educated at the University of St Petersburg. After working at Yurev University from 1902 to 1913, he returned to St Petersburg where he spent the rest of his career. He studied the mechanics of solid bodies and the theory of elasticity, particularly the complex variable theory. In 1907 Kolosov derived the solution for stresses around an elliptical hole. It showed that the concentration of stress became far greater as the radius of curvature at an end becomes small compared with the overall length of the hole. Engineers needed to understand the issue of stress concentration in order to keep their design safe [128].

7. Pre-electronic computer era

Numerical efforts solving boundary value problems predate the emergence of digital computers. One important contribution is the Ritz method, proposed by *Walter Ritz* (1878–1909) in 1908 [140]. When applied to subdomains, the Ritz method is considered to be the forerunner of the Finite Element Method [194]. Ritz' idea involves the use of variational method and trial functions to find approximate solutions of boundary value problems. For example, for the following functional

$$\Pi = \iiint_{\Omega} \frac{1}{2}(\nabla\phi)^2 \mathrm{d}V - \iint_{\Gamma} \frac{\partial\phi}{\partial n}(\phi - f)\mathrm{d}S \tag{57}$$

finding its stationary value by variational method leads to

$$\delta\Pi = -\iiint_{\Omega} \delta\phi\nabla^2\phi \, \mathrm{d}V - \iint_{\Gamma} \delta\left(\frac{\partial\phi}{\partial n}\right)(\phi - f)\mathrm{d}S = 0 \tag{58}$$

Since the variation is arbitrary, the above equation is equivalent to the statement of Dirichlet problem

$$\nabla^2\phi = 0 \quad \text{in } \Omega \tag{59}$$

and

$$\phi = f(\mathbf{x}) \quad \text{on } \Gamma \tag{60}$$

Ritz proposed to approximate ϕ using a set of trial functions ψ_i by the finite series

$$\phi \approx \sum_{i=1}^{n} \alpha_i \psi_i \tag{61}$$

where α_i are constant coefficients to be determined. Eq. (61) is substituted into the functional (57) and the variation is taken with respect to the n unknown coefficients a_i. The domain and boundary integration were performed, often in the subdomains, to produce numerical values. This leads to a linear system that can be solved for α_i. The above procedure involves the integration over the solution domain; hence it is considered as a domain method, not a boundary method.

Based on the same idea, *Erich Trefftz* (1888–1937) in his 1926 article 'A counterpart to Ritz method' [171,172] devised the boundary method, known as the Trefftz method. Utilizing Green's first identity (23), we can write (57) in an alternate form

$$\Pi = -\iiint_{\Omega} \frac{1}{2}\phi\nabla^2\phi \, \mathrm{d}V - \iint_{\Gamma} \frac{\partial\phi}{\partial n}\left(\frac{1}{2}\phi - f\right)\mathrm{d}S \tag{62}$$

In making the approximation (61), Trefftz proposed to use trial functions ψ_i that satisfy the governing differential equation

$$\nabla^2\psi_i = 0 \tag{63}$$

but not necessarily the boundary condition. For Laplace equation, these could be the harmonic polynomials

$$\psi_i = \{1, x, y, z, x^2 - y^2, y^2 - z^2, z^2 - x^2, xy, yz, \dots\} \tag{64}$$

With the substitution of (61) into (62), the domain integral vanishes, and the functional is approximated as

$$\Pi \approx -\iint_\Gamma \sum_{i=1}^n \alpha_i \frac{\partial\psi_i}{\partial n}\left(\frac{1}{2}\sum_{i=1}^n \alpha_i\psi_i - f\right)\mathrm{d}S \tag{65}$$

Taking variation of (65) with respect to the undetermined coefficients α_j, and setting each part associated with the variations $\delta\alpha_j$ to zero, we obtain the linear system

$$\sum_{i=1}^n a_{ij}\alpha_i = b_j; \quad j = 1, \dots, n \tag{66}$$

where

$$a_{ij} = \frac{1}{2}\iint_\Gamma \frac{\partial\psi_i\psi_j}{\partial n}\mathrm{d}S \tag{67}$$

$$b_j = \iint_\Gamma f\frac{\partial\psi_j}{\partial n}\mathrm{d}S \tag{68}$$

Eq. (66) can be solved for α_i.

The above procedure requires the integration of functions over the solution boundary. In the present day Trefftz method, a simpler procedure is often taken. Rather than minimizing the functional over the whole boundary, one can enforce the boundary condition on a finite set of boundary points $\mathbf{x}_j$ such that

$$\phi(\mathbf{x}_j) \approx \sum_{i=1}^n \alpha_i\psi_i(\mathbf{x}_j) = f(\mathbf{x}_j); \quad j = 1, \dots, n \text{ and } \mathbf{x}_j \in \Gamma \tag{69}$$

This is a point collocation method and there is no integration involved. Eq. (69) can also be derived from a weighted residual formulation using Dirac delta function as the test function.

Following the same spirit of the Trefftz method, one can use the fundamental solution as the trial function. Since fundamental solution satisfies the governing equation as

$$\mathfrak{L}\{G(\mathbf{x}, \mathbf{x}')\} = \delta(\mathbf{x}, \mathbf{x}') \tag{70}$$

where $\mathfrak{L}$ is a linear partial differential operator, G is the fundamental solution of that operator, and δ is the Dirac delta function. It is obvious that the approximate solution

$$\phi(\mathbf{x}) \approx \sum_{i=1}^n \alpha_i G(\mathbf{x}, \mathbf{x}_i); \quad \mathbf{x} \in \Omega, \quad \mathbf{x}_i \notin \Omega \tag{71}$$

satisfies the governing equation as long as the source points $\mathbf{x}_i$ are placed outside of the domain. To ensure that the boundary condition is satisfied, again the point collocation is applied:

$$\phi(\mathbf{x}_j) \approx \sum_{i=1}^n \alpha_i G(\mathbf{x}_j, \mathbf{x}_i) = f(\mathbf{x}_j); \quad j = 1, \dots, n \text{ and } \mathbf{x}_j \in \Gamma \tag{72}$$

This is called the method of fundamental solutions.

The superposition of fundamental solutions is a well known solution technique in fluid mechanics for exterior domain problems. *William John Macquorn Rankine* (1820–1872) in 1864 [135] showed that the superposition of sources and sinks along an axis, combining with a rectilinear flow, created the field of uniform flow around closed bodies, known as Rankine bodies. Various combinations were experimented to create different body shapes. However, there was no direct control over the shape. It took *Theodore von Kármán* (1881–1963) in 1927 [184] to propose a collocation procedure to create the arbitrarily desirable body shapes. He distributed $n+1$ sources and sinks of unknown strengths along the axis of an axisymmetric body, adding to a rectilinear flow

$$\phi(\mathbf{x}) \approx Ux + \sum_{i=1}^{n+1}\frac{\sigma_i}{4\pi r(\mathbf{x}, \mathbf{x}_i)} \tag{73}$$

where U is the uniform flow velocity, $\mathbf{x}_i$ are points on x-axis, and σ_i are source/sink strengths. The strengths can be determined by forcing the normal flux to vanish at n specified points on the meridional trace of the axisymmetric body. An auxiliary condition

$$\sum_{i=1}^{n+1}\sigma_i = 0 \tag{74}$$

is needed to ensure the closure of the body. In fact, other singularities, such as doublets (dipoles) and vortices, can be distributed inside a body to create flow around arbitrarily shaped two- and three-dimensional bodies [174].

In 1930 von Kármán [185] further proposed the distribution of singularity along a line inside a two-dimensional streamlined body to generate the potential

$$\phi(\mathbf{x}) = -\int_L \ln r(\mathbf{x}, \xi)\sigma(\xi)\mathrm{d}s(\xi); \quad \mathbf{x} \in \Omega_e \tag{75}$$

where ϕ is the perturbed potential from the uniform flow field, σ is the distribution density, L is a line inside the body, and Ω_e is the external domain (Fig. 3a). For vanishing potential at infinity, the following auxiliary condition is needed

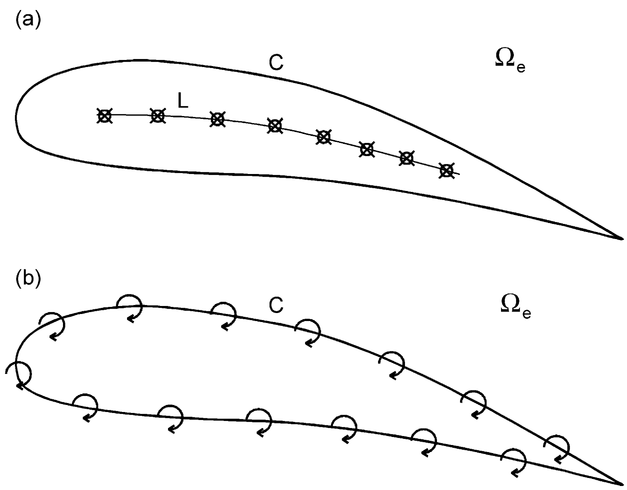


Fig. 3. Two methods of distributing sigularities: (a) sources, sinks, and doublets on a line L inside the airfoil; (b) vortices on the surface C of the airfoil.

$$\int_L \sigma(\xi)\mathrm{d}s(\xi) = 0 \tag{76}$$

To find the distribution density, Neumann boundary condition is enforced on a set of discrete points $\mathbf{x}_i$, $i = 1,\ldots n$, on the surface of the body

$$\frac{\partial\phi(\mathbf{x}_i)}{\partial n(\mathbf{x}_i)} = -\int_L \frac{\partial \ln r(\mathbf{x}_i, \xi)}{\partial n(\mathbf{x}_i)}\sigma(\xi)\mathrm{d}s(\xi); \quad \mathbf{x}_i \in C \tag{77}$$

where C is the boundary contour (Fig. 3a).

Prager [134] in 1928 proposed a different idea: vortices are distributed on the surface of a streamlined body (Fig. 3b) to generate the desirable potential. When this is written in terms of stream function ψ, the integral equation becomes

$$\psi(\mathbf{x}) = -\int_C \ln r(\mathbf{x}, \xi)\sigma(\xi)\mathrm{d}s(\xi); \quad \mathbf{x} \in \Omega_\mathrm{e} \tag{78}$$

In this case, Dirichlet condition is enforced on the surface of the body.

Lotz [113] in 1932 proposed the discretization of Fredholm integral equation of the second kind on the surface of an axisymmetric body for solving external flow problems. The method was further developed by Vandrey [177,178] in 1951 and 1960. Other early efforts in solving potential flows around obstacles, prior to the invention of electronic computers, can be found in a review [90].

In 1937 Muskhelishvili [124] derived the complex variable equations for elasticity and suggested to solve them numerically. The actual numerical implementation was accomplished in 1940 by Gorgidze and Rukhadze [78] in a procedure that resembled the present-day BEM: it divided the contour into elements, approximated the function within the elements, and formed a linear algebraic system consisting the unknown coefficients.

The above review demonstrates that finding approximate solutions of boundary value problems using boundary or boundary-like discretization is not a new idea. These early attempts of Trefftz, von Kármán, and Muskhelishvili existed before the electronic computers. However, despite these heroic attempts, without the aid of modern computing tools these calculations had to be performed by human or mechanical computers. The drudgery of computation was a hindrance for their further development; hence these methods remained dormant for a while and had to wait for a later date to be rediscovered.

7.1. Ritz

Walter Ritz (1878–1909) was born in Sion in the southern Swiss canton of Valais. As a specially gifted student, the young Ritz excelled academically at the Lycèe communal of Sion. In 1897 he entered the Polytechnic School of Zurich where he began studies in engineering. He soon found that he could not live with the approximations and compromises involved with engineering, so he switched to the more mathematically exacting studies in physics, where *Albert Einstein* (1879–1955) was one of his classmates. In 1901 he transferred to Göttingen, where his rising aspirations were strongly influenced by *Woldemar Voigt* (1850–1919) and Hilbert. Ritz's dissertation on spectroscopic theory led to what is known as the Ritz combination principle. In the next few years he continued his work on radiation, magnetism, electrodynamics, and variational method. But in 1904 his health failed and he returned to Zurich. During the following 3 years, Ritz unsuccessfully tried to regain his health and was outside the scientific centers. In 1908 he relocated to Göttingen where he qualified as a Privat Dozent. There he produced his opus magnum *Recherches critiques sur l'Électrodynamique Générale*. In 1908–1909 Ritz and Einstein held a war in *Physikalische Zeitschrift* over the proper way to mathematically represent black-body radiation and over the theoretical origin of the second law of thermodynamics. The debated was judged to Ritz's favor. Six weeks after the publication of this series, Ritz died at the age of 31, leaving behind a short but brilliant career in physics [69].

7.2. von Kármán

Theodore von Kármán (1881–1963) was born in Budapest, Hungary. He was trained as a mechanical engineer in Budapest and graduated in 1902. He did further graduate studies at Göttingen and earned his PhD in 1908 under *Ludwig Prandtl* (1875–1953). In 1911 he made an analysis of the alternating double row of vortices behind a bluff in a fluid stream, known as Kármán's vortex street. In 1912, at the age of 31, he became Professor and Director of Aeronautical Institute at Aachen, where he built the world's first wind tunnel. In World War I, he was called into military service for the Austro-Hungarian Empire and became Head of Research in the Air Force, where he led the effort to build the first helicopter.

After the war, he was instrumental in calling an International Congress on Aerodynamics and Hydrodynamics at Innsbruck, Austria, in 1922. This meeting became the forerunner of the International Union of Theoretical and Applied Mechanics (IUTAM) with von Kármán as its honorary president. He first visited the United States in 1926. In 1930 he headed the Guggenheim Aeronautical Lab at the California Institute of Technology. In 1944, he cofounded of the present NASA Jet Propulsion Laboratory and undertook America's first governmental long-range missile and space-exploration research program. His personal scientific work included contributions to fluid mechanics, turbulence theory, supersonic flight, mathematics in engineering, and aircraft structures. He is widely recognized as the father of modern aerospace science [186].

7.3. Trefftz

Erich Trefftz (1888–1937) was born on February 21, 1888 in Leipzig, Germany. In 1890, the family moved to Aachen. In 1906 he began his studies in mechanical engineering at the Technical University of Aachen, but soon changed to mathematics. In 1908 Trefftz transferred to Göttingen, at that time the Mecca of mathematics and physics. Here after Gauss, Dirichlet, and Riemann, now Hilbert, *Felix Christian Klein* (1849–1925), *Carle David Tolmé Runge* (1856–1927), and Prandtl created a continuous progress of first-class mathematics. Trefftz's most important teachers were Runge, Hilbert and also Prandtl, the genius mechanician in modern fluid- and aero-dynamics. Trefftz spent 1 year at the Columbia University, New York, and then left Göttingen for Strassburg to study under the guidance of the famous Austrian applied mathematician *Richard von Mises* (1883–1953), who founded the GAMM (Gesellschaft für Angewandte Mathematik und Mechanik) in 1922 together with Prandtl. Mises was also the first editor of ZAMM (Zeitschrift für Angewandte Mathematik und Mechanik).

Trefftz's academic career began with his doctoral thesis in Strassburg in 1913, where he solved a mathematical problem of hydrodynamics. He was a soldier in the first World War, but already in 1919 he got his habilitation and became a Full Professor of Mathematics in Aachen. In the year 1922 he got a call as a Full Professor with a Chair in the Faculty of Mechanical Engineering at the Technical University of Dresden. There he became responsible for teaching and research in strength of materials, theory of elasticity, hydrodynamics, aerodynamics and aeronautics. In 1927 he moved from the engineering to the mathematical and natural science faculty, being appointed there as a Chair in Technical (Applied) Mechanics.

Trefftz had a lifelong friendship with von Mises, who, being a Jew, had to leave Germany in 1933. Trefftz felt and showed outgoing solidarity and friendship to von Mises, and he clearly was in expressed distance to the Hitler regime until his early death in 1937. Feeling the responsibility for science, he took over the Presidentship of GAMM, and became the Editor of ZAMM in 1933 in full accordance with von Mises [160].

7.4. Muskhelishvili

Nikolai Ivanovich Muskhelishvili (1891–1976) was a student at the University of St Petersburg. He was naturally influenced by the glorious tradition of the St Petersburg mathematical school, which began with Euler and continued by the prominent mathematicians such as Ostrogradsky, *Pafnuty Lvovich Chebyshev* (1821–1894), and *Aleksandr Mikhailovich Lyapunov* (1857–1918). As an undergraduate student, Muskhelishvili was greatly impressed by the lectures of Kolosov on the complex variable theory of elasticity. Muskhelishvili took this topic as his graduation thesis and performed brilliantly that Kolosov decided to publish these results as a coauthor with his student in 1915. In 1922 Muskhelishvili became a Professor at the Tbilisi State University, where he remained until his death. In 1935 he published the masterpiece *Some Basic Problems of the Mathematical Theory of Elasticity*, which won him the Stalin Prize of the First Degree. He held many positions such as Chair, Director, President, at Tbilisi and the Georgian Branch of USSR Academy of Sciences, and received many honors [196,197].

8. Electronic computer era

Although electronic computers were invented in the 1940s, they did not become widely available to common researchers until the early 1960s. It is not surprising that the development of the finite element method [38], as well as a number of other numerical methods, started around that time. A number of independent efforts of experimenting on boundary methods also emerged in the early 1960s. Some of the more significant ones are reviewed below.

Friedman and Shaw [68] in 1962 solved the scalar wave equation

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{79}$$

in the time domain for the scattered wave field resulting from a shock wave impinging on a cylindrical obstacle. In the above ϕ is the velocity potential and c is the wave speed. The use of the fundamental solution

$$G = -\frac{1}{r}\delta\left[\frac{r}{c} - (t - t_0)\right] \tag{80}$$

where δ is the Dirac delta function, in Green's second identity (24) produced the boundary integral equation

$$\phi_{sc}(\mathbf{x}, t) = \frac{1}{4\pi}\int_0^{t^+}\iint_\Gamma \left[G\frac{\partial\phi_{sc}}{\partial n} - \phi_{sc}\frac{\partial G}{\partial n}\right]\mathrm{d}S\,\mathrm{d}t_0 \tag{81}$$

where ϕ_{sc} is the scattered wave field. Eq. (81) was further differentiated with respect to time to create the equation for acoustic pressure. For a two-dimensional problem, the equation was discretized in space (boundary contour) and in time that resulted into a double summation. Variables were assumed to be constant over space and time subintervals, so the integration could be performed exactly. Finite difference explicit time-stepping scheme was used and the resultant algebraic system required only successive, not simultaneous solution. The computation was carried out using a Monroe desk calculator [154]. The scattering due to a box-shaped rigid obstacle was solved. The work was extended in 1967 by Shaw [152] to handle different boundary conditions on the obstacle surface, and by Mitzner [122] using the retarded potential integral representation.

Banaugh and Goldsmith [5] in 1963 tackled the two-dimensional wave equation in the frequency domain, governed by the Helmholtz equation (38). The 2-D boundary integral equation counterpart to the 3-D version (40) is

$$\phi = \frac{i}{4}\int_C \left[H_0^{(1)}(kr)\frac{\partial\phi}{\partial n} - \phi\frac{\partial H_0^{(1)}(kr)}{\partial n}\right]\mathrm{d}s \tag{82}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero, k is the wave number, and $i = \sqrt{-1}$. Eq. (82) is solved in the complex variable domain. Similar to Friedman and Shaw [68], the integration over a subinterval was made easy by assuming constant variation of the potential on the subinterval. The problem of a steady state wave scattered from the surface of a circular cylinder was solved as a demonstration. An IBM 7090 mainframe computer was used for the numerical solution. It is of interest to observe that the discretization was restricted to 36 points, corresponding to 72 unknowns (real and imaginary parts), due to the memory restriction of the computer. A larger linear system would have required the read/write operation on the tape storage and special linear system solution algorithm.

In the same year (1963) Chen and Schweikert [36] solved the three-dimensional sound radiation problem in the frequency domain using the Fredholm integral equation of the second kind

$$\frac{\partial\phi(\mathbf{x})}{\partial n(\mathbf{x})} = -2\pi\sigma(\mathbf{x}) + \iint_\Gamma \sigma(\xi)\frac{\partial}{\partial n(\mathbf{x})}\left(\frac{e^{ikr}}{r}\right)\mathrm{d}S(\xi); \quad \mathbf{x}\in\Gamma \tag{83}$$

Problems of vibrating spherical and cylindrical shells in infinite fluid domain were solved. The surface was divided into triangular elements. An IBM 704 mainframe was used, which allowed up to 1000 degrees of freedom to be modeled.

Subsequent work using integral equation solving acoustic scattering problems included Chertock [37] in 1964, and Copley in 1967 [40] and 1968 [41]. Copley was the first to report the non-uniqueness of integral equation formulation due to the existence of eigen frequencies. Schenck [150] in 1968 presented the CHIEF (combined Helmholtz integral equation formulation). Waterman developed the null-field and T-matrix method, first in 1965 [188] for solving electromagnetic scattering problems, and then in 1969 [189] for acoustic problems. In both the CHIEF and the T-matrix method, the so-called 'null-field integral equation' or the 'interior Helmholtz integral equation' was utilized:

$$0 = \iint_\Gamma \phi(\xi)\frac{\partial G(\mathbf{x},\xi)}{\partial n(\xi)} - G(\mathbf{x},\xi)\frac{\partial\phi(\xi)}{\partial n(\xi)}\mathrm{d}S(\xi); \quad \mathbf{x}\in\Omega_i \tag{84}$$

where Ω_i is the interior of the scatterer, and

$$G = -\frac{1}{4\pi r}e^{-ikr} \tag{85}$$

is the free-space Green's function of Helmholtz equation. The left hand side of (84) is null because the source point $\mathbf{x}$ is placed inside the body, which is outside the wave field. The above equation was combined with the exterior integral equation to eliminate the non-unique solution, or the so-called 'spurious frequencies'.

Returning to potential problems, *Maurice Aaron Jaswon* (1922-) and Ponter [95] in 1963 employed Green's third identity (25) for the numerical solution of prismatic bars subjected to torsion in two dimensions,

$$\phi(\mathbf{x}) = \frac{1}{\pi}\int_C \left[\phi(\xi)\frac{\partial \ln r(\mathbf{x},\xi)}{\partial n(\xi)} - \ln r(\mathbf{x},\xi)\frac{\partial\phi(\xi)}{\partial n(\xi)}\right]\mathrm{d}s(\xi); \quad \mathbf{x}\in C \tag{86}$$

The boundary conditions were Dirichlet type. Ponter [133] in 1966 extended it to multiple domain problems.

Jaswon [97] and Symm [164] in 1963 used the single-layer method, i.e. the Fredholm equation of the first kind as shown in (33), but in two dimensions,

$$\phi(\mathbf{x}) = -\int_C \ln r(\mathbf{x},\xi)\,\sigma(\xi)\mathrm{d}s(\xi); \quad \mathbf{x}\in C \tag{87}$$

for the solution of Dirichlet problems. The above equation was supposedly to be unstable. However, apparently good solutions were obtained. For Neumann problems, the Fredholm integral equation of the second kind

$$\frac{\partial\phi(\mathbf{x})}{\partial n(\mathbf{x})} = \pi\sigma(\mathbf{x}) - \int_C \frac{\partial \ln r(\mathbf{x},\xi)}{\partial n(\mathbf{x})}\sigma(\xi)\mathrm{d}s(\xi); \quad \mathbf{x}\in C \tag{88}$$

was used. In the same paper [164], a mixed boundary value problem was solved using Green's formula (86), rather than the Fredholm integral equations.

Hess [89] in 1962 and Hess and Smith [91] in 1964 utilized the single-layer method (30) to solve problems of external potential flow around arbitrary three-dimensional bodies

$$\frac{\partial\phi(\mathbf{x})}{\partial n(\mathbf{x})} = -2\pi\sigma(\mathbf{x}) + \iint_\Gamma \frac{\partial 1/r(\mathbf{x},\xi)}{\partial n(\mathbf{x})}\sigma(\xi)\mathrm{d}s(\xi); \quad \mathbf{x}\in\Gamma \tag{89}$$

The formulation was the same as that of Lotz [113] and Vandrey [177,178]. The surface of the body was discretized into quadrilateral elements and the source density was assumed to be constant on the element. This technique, called the surface source method, has been developed into a powerful numerical tool for the aircraft industry [90].

Massonet [115] in 1965 discussed a number of ideas of using boundary integral equations solving elasticity problems. Numerical solutions were carried out in two cases. In the first case, Fredholm integral equation of the second kind was used to solve torsion problems:

$$\phi(\mathbf{x}) = -\pi\mu(\mathbf{x}) - \int_C \frac{\partial \ln r(\mathbf{x},\xi)}{\partial n(\xi)}\mu(\xi)\mathrm{d}s(\xi); \quad \mathbf{x}\in C \tag{90}$$

In the second case, plane elasticity problems were solved using the distribution of the radial stress field resulting from a half-plane point force on the boundary. The following Fredholm equation of the second kind was used:

$$\mathbf{t}(\mathbf{x}) = \boldsymbol{\mu}(\mathbf{x}) - \frac{2}{\pi}\int_C \boldsymbol{\mu}(\xi)\frac{\cos\varphi\cos\alpha}{r}\mathbf{e}_r\,\mathrm{d}s(\xi); \quad \mathbf{x}\in C \tag{91}$$

where $\mathbf{t}$ is the boundary traction vector, $\boldsymbol{\mu}$ is the intensity of the fictitious stress, μ is its magnitude, $\mathbf{e}_r$ is the unit vector in the r direction, ϕ is the angle between the two vectors $\boldsymbol{\mu}$ and $\mathbf{e}_r$, and α is the angle between $\mathbf{e}_r$ and the boundary normal. Solution were found using the iterative procedure of successively approximating the function $\boldsymbol{\mu}$. Due to the half-plane kernel function used, this technique applies only to simply-connected domains.

During the first decade of the 20th century, the introduction of the Fredholm integral equation theorem put the potential theory on a solid foundation. Although attempted by Fredholm himself, the same level of success was not found for elasticity problems. In fact, similar rigorousness was not accomplished for another 40 years [72]. Started in the 1940s, the Georgian school of elasticians, led by Muskhelishvili [196,197] and followed by *Ilia Nestorovich Vekua* (1907–1977) [179], *Nikolai Petrovich Vekua* (1913–1993) [180], and *Victor Dmitrievich Kupradze* (1903–1985) [104,106], all associated with the Tbilisi State University, together with *Solomon Grigorevich Mikhlin* (1908–1991) [120] of St Petersburg, made important progresses in the theory of vector potentials (elasticity) through the study of singular integral equations. The initial development, however, was limited to one-dimensional singular integral equations, which solved only two-dimensional problems. The development of multi-dimensional integral equations started in the 1960s [72].

Kupradze in 1964 [105] and 1965 [104] discussed a method for finding approximate solutions of potential and elasticity static and dynamic problems. He called the approach 'method of functional equations'. Numerical examples were given in two dimensions. For potential problems with the Dirichlet boundary condition

$$\phi = f(\mathbf{x}), \quad \mathbf{x}\in C \tag{92}$$

where C is the boundary contour, the solution is represented by the pair of integral equations

$$\phi(\mathbf{x}) = \frac{1}{2\pi}\int_C f(\xi)\frac{\partial \ln r(\mathbf{x},\xi)}{\partial n(\xi)}\mathrm{d}s(\xi) + \frac{1}{\pi}\int_C \sigma(\xi)\ln r(\mathbf{x},\xi)\mathrm{d}s(\xi); \quad \mathbf{x}\in\Omega \tag{93}$$

$$0 = \frac{1}{2\pi}\int_C f(\xi)\frac{\partial \ln r(\mathbf{x},\xi)}{\partial n(\xi)}\mathrm{d}s(\xi) + \frac{1}{\pi}\int_C \sigma(\xi)\ln r(x,\xi)\mathrm{d}s(\xi); \quad \mathbf{x}\in C' \tag{94}$$

In the above C' is an arbitrary auxiliary boundary that encloses C, and σ is the distribution density, which needs to be solved from (94). We notice that the above equations involve the distribution of both the single-layer and the double-layer potential. Another observation is that in (94) the center of singularity $\mathbf{x}$ is located on C', which is *outside* of the solution domain Ω. Since C and C' are distinct contours, Eq. (94) is not an integral equation in the classical sense, in which the singularities are located on the boundary. This is why the term 'functional equation' was used instead. In the numerical implementation, C' was chosen as a circles, upon which n nodes were selected to place the singularity. Since the singularities were not located on the boundary C, the integrals in (94) were regular and could be numerically evaluated using a simple quadrature rule. Gaussian quadrature with n nodes was used for the integration. The resultant linear system was solved for the n discrete σ values located at the quadrature nodes. Eq. (93) was then used to find solution at any point in the domain.

For elasticity problems, the same technique was employed. For static, two-dimensional problems with prescribed boundary displacement

$$u_i = f_i(\mathbf{x}), \quad \mathbf{x} \in C \tag{95}$$

the following pair of vector integral equations solve the boundary value problem:

$$u_j(\mathbf{x}) = \frac{1}{\pi} \int_C \sigma_i(\xi) u_{ij}^f(\mathbf{x}, \xi) \mathrm{d}s(\xi) - \frac{1}{2\pi} \int_C f_i(\xi) u_{ij}^d(\mathbf{x}, \xi) \mathrm{d}s(\xi); \quad \mathbf{x} \in \Omega \tag{96}$$

$$0 = \frac{1}{\pi} \int_C \sigma_i(\xi) u_{ij}^f(\mathbf{x}, \xi) \mathrm{d}s(\xi) - \frac{1}{2\pi} \int_C f_i(\xi) u_{ij}^d(\mathbf{x}, \xi) \mathrm{d}s(\xi); \quad \mathbf{x} \in C' \tag{97}$$

where σ_i is the distribution density (vector), and the kernel

$$u_{ij}^f = \frac{1}{4G(1-\nu)} \left[(3-4\nu)\delta_{ij} \ln r - \frac{x_i x_j}{r^2} \right] \tag{98}$$

is the fundamental solution due to a point force (single-layer potential) in the x_j direction, and

$$u_{ij}^d = \frac{1}{2(1-\nu)} \times \frac{1}{r} \left[(1-2\nu) \left(\frac{n_i x_j}{r} - \frac{n_j x_i}{r} + \delta_{ij} \frac{\partial r}{\partial n} \right) + 2 \frac{x_i x_j}{r^2} \frac{\partial r}{\partial n} \right] \tag{99}$$

is the fundamental solution due to a dislocation (double-layer potential) oriented in the x_j direction. Kupradze's method was closely followed in Russia under the name 'potential method', particularly in the solution of shells [71,176] and plates [103,181].

Kupradze's technique of distributing fundamental solutions on an exterior, auxiliary boundary has been considered by some as the origin of the 'method of fundamental solutions' [14]. However, in most applications [65], the method of fundamental solutions often bypasses the integral equation formulation. It considers the distribution of admissible solutions of discrete and unknown density on an external auxiliary boundary, for example, in the form of (71). The boundary conditions are satisfied by collocating at a set of boundary nodes. Hence the method of fundamental solutions, although can be viewed as a special case of the method of functional equations, was in fact independently developed. For example, Oliveira [130] in 1968 proposed the use of fundamental solution of point forces linearly distributed over linear segments to solve plane elasticity problems. For potential problems, the recent origin can be traced to Mathon and Johnston [116] in 1977.

Another type of problems that has traditionally used boundary methods involves problems with discontinuities such as fractures, dislocations resulting from imperfections of crystalline structures, interface between dissimilar materials, and other discontinuities. In these cases, certain physical quantities, such as displacements or stresses, suffer a jump. These discontinuities can be simulated by the distribution of singular solutions such as the Volterra [183] and Somigliana dislocations [158,159] over the physical surface, which often results in integral equations [12,62]. For example, integral equation of this type

$$A(x)\psi(x) + \frac{1}{\pi} \int_a^b \frac{B(x')\psi(x')}{x'-x} \mathrm{d}x' + \int_a^b K(x,x')\psi(x') \mathrm{d}x' = f(x), \quad a < x < b \tag{100}$$

and other types were numerically investigated by Erdogan and Gupta [60,61] in 1972 using Chebyshev and Jacobi polynomials for the approximation. These type of one-dimensional singular integral equations has also been solved using piece-wise, low degree polynomials [74].

The review presented so far has focused on the solution of physical and engineering problems. The formulations often borrow the physical idea of distributing concentrated loads; hence the integral equations are typically singular. Due to the existence of multiple spatial and the time dimensions in physical problems, the integral equations are often multi-dimensional.

For the mathematical community, the effort of finding approximate solutions of integral equations existed since the major breakthrough of Fredholm in the 1900s. Early efforts focused on finding successive approximations of linear, one-dimensional, and non-singular integral equations. Different kinds of integral equations that may or may not have physical origin were investigated. One of the first monographs on numerical solution of integral equations is by Bückner [29] in 1952. Another early monograph is by Mikhlin and Smolitsky [121] in 1967. The field flourished in the 1970 with the publication of several monographs—Kagiwada and Kalaba [98] in 1974, Atkinson [2] in 1976, Ivanov [92] in 1976, and Baker [4] in 1977. As mentioned above, mostly one-dimensional integral equations were investigated. Some integral equations have physical origin such as flow around hydrofoil, population competition, and quantum scattering [55], while most others do not. The methods used included projection method, polynomial collocation, Galerkin method, least squares, quadrature method, among others [77]. It is of interest to observe that the developments in the two communities, the applied mathematics and the engineering, run parallel to each other, almost devoid of cross citations, although it is clear that cross-fertilization will be beneficial.

As seen from the review above, the origins of boundary numerical methods, as well as many other numerical methods, can be traced to this period, during which many ideas sprouted. However, even though methods like those by Jaswon and Kupradze started to receive attention, these efforts did not immediately coalesce into a single 'movement' that grows rapidly. In the following sections we shall review those significant events that led to the development of the modern-day boundary integral equation method and the boundary element method.

8.1. Kupradze

Viktor Dmitrievich Kupradze (1903–1985) was born in the village of Kela, Russian Georgia. He was enrolled in the Tbilisi State University in 1922 and was awarded the diploma in mathematics in 1927. He stayed on as a Lecturer in mathematical analysis and mechanics until 1930. In that year he entered the Steklov Mathematical Institute in Leningrad for postgraduate study and obtained his doctor of mathematics degree in 1935. In 1933 Muskhelishvili founded a research institute of mathematics, physics and mechanics in Tbilisi. In 1935, Muskhelishvili and his closest associates Kupradze and Vekua transformed the institute and became affiliated with the Georgian Academy of Sciences, with Kupradze serving as its first director from 1935 to 1941. The Institute was later known as A. Razmadze. From 1937 until his death, Kupradze served as the Head of the Differential and Integral Equations Department at Tbilisi. Kupradze's research interest covered the theory of partial differential equations and integral equations, and mathematical theory of elasticity and thermoelasticity. He received many honors, including political ones. He was elected as an Academician of the Georgian Academy in 1946. From 1954 to 1958 he served as the Rector of Tbilisi University, and from 1954 to 1963 the Chairman of Supreme Soviet of the Georgian SSR.

8.2. Jaswon

Maurice Aaron Jaswon (1922-) was born in Dublin, Ireland. He was enrolled in the Trinity College, Dublin, and obtained his BSc degree in 1944. He entered the University of Birmingham, UK and was awarded his PhD degree in 1949. In the same year he started his academic career as a Lecturer in Mathematics at the Imperial College, London. His early research was focused on the mathematical theory of crystallography and dislocation, which cumulated into a book published in 1965 [94], with a updated version in 1983 [96]. In 1957 Jaswon was promoted to the Reader position and stayed at the Imperial College until 1967. It was during this period that he started his seminal work on numerical solution of integral equations with his students George Thomas Symm [165] and Alan R.S. Ponter. In 1963–1964 Jaswon visited Brown University. In 1965–1966 he was a visiting Professor at the University of Kentucky. His presence there was what initially made Frank Rizzo aware of an opening position at Kentucky [143]. Upon Rizzo's arrival in 1966, they had a few months of overlapping before Jaswon's returning to England. In 1967 Jaswon left the Imperial College to take a position as Professor and Head of Mathematics at the City University of London, where he stayed for the next 20 years until his retirement in 1987. He remains active as an Emeritus Professor at City University. Jaswon was considered by some as the founder of the boundary integral equation method based on his 1963 work [95] implementing Green's formula.

9. Boundary integral equation method

A turning point marking the rapid growth of numerical solutions of boundary integral equations happened in 1967, when *Frank Joseph Rizzo* (1938-) published the article 'An integral equation approach to boundary value problems of classical elastostatics' [142]. In this paper, a numerical procedure was applied for solving the Somigliana identity (45) for elastostatics problems. The work was an extension of Rizzo's doctoral dissertation [141] at the University of Illinois, Urbana-Champaign, which described the numerical algorithm, yet without actual implementation.

According to Rizzo's own recollection [145], he was deeply influenced by his advisor Marvin Stippes. At that time, Stippes was studying representation integrals for elastic field in terms of boundary data. The Somigliana identity received particular attention. The identity (45) without the body force can be written as follows:

$$u_j = \iint_\Gamma (t_i u_{ij}^* - t_{ij}^* u_i)\mathrm{d}S \tag{101}$$

Although the above equation appears to give the solution of the displacement field, it actually does not, as the right hand side contains unknowns. In a well-posed boundary value problem, only half of the boundary data pair $\{t_i, u_i\}$ is given. The question is whether it is possible to exploit the above equation by using the then-new crop of digital computers to do arithmetic and to develop a systematic solution process.

While Rizzo was struggling with these ideas, Stippes called to his attention the recently published papers by Jaswon [95,97,164] in which numerical solutions of potential problems were attempted by exploiting Green's third identity. By realizing that the Somigliana identity is just the vector version of the potential theory, in Rizzo's own words: [145] 'a 'light bulb' appeared in my consciousness'. Rizzo further stated: [143] 'That work on potential theory was the model, motivation, and springboard for everything I did that year for elasticity theory…. Indeed, in retrospect, all three of those papers represent at once the birth and quintessence of what has become known as the 'direct' boundary element or boundary integral equation method…'

Rizzo's subsequent implementation of numerical solution can be viewed from the angle of *Thomas Allen Cruse* (1941-): [51] '[In 1965] I took a leave of absence from Boeing and enrolled in the Engineering Mechanics program at the University of Washington. One of the first faculty members I met was Frank Rizzo who had just completed his doctoral studies at the University of Illinois in Urbana and [in 1964] had come to the University of Washington as an Assistant Professor in the Civil Engineering Department. When I met Professor Rizzo he was working with a graduate student [C.C. Chang] who could program in Fortran. Frank's original dissertation was the formulation of the two-dimensional elasticity problem using Betti's reciprocal work theorem. The Quarterly of Applied Mathematics rejected the manuscript derived from Frank's dissertation due to the absence of numerical results!' Rizzo developed the algorithm and obtained good numerical results; and the paper was finally published in that journal in 1967 [142].

Cruse continued to reminisce about his own role in this early development of BEM: [51] 'I was enrolled after this time in an elastic wave propagation course taught by Professor Rizzo. As one of his graduate students I was searching for a suitable piece of original research upon which to base my dissertation. One day, Frank showed us how the Laplace transform converted the hyperbolic wave equation into an elliptic equation—I found my research topic at that point'.

After planting the seed that became Cruse's doctoral research [144], Rizzo moved to the University of Kentucky in 1966. Cruse completed his dissertation independently in 1967 [43]. In Kentucky, Rizzo met David J. Shippy and started a highly productive collaboration. In Shippy's recollection [156]: 'When Frank became a faculty member at the University of Kentucky in 1966 he had already written his seminal paper on the use of integral equations to solve elasticity problems [142]. Before long, Frank discovered that I was very much interested in and involved with the use of computers. He approached me, described his research interest, and proposed that we collaborate on future research of this kind… With no computer code in hand, Frank and I proceeded to develop from scratch some ad hoc (for specific geometries) direct boundary integral equation code for solution of plane elastostatics problems…. Having that small success behind us, we were poised to apply the boundary integral equation (BIE) method to more difficult problems'.

Their first try was to solve elasticity problems with inclusions [146]. Next they tackled plane anisotropic bodies [147]. Utilizing Laplace transform and the numerical Laplace inversion, Rizzo and Shippy then solved the transient heat conduction problems [148] and the quasi-static viscoelasticity problems [149]. Hence in a quick succession of work from 1968 to 1971, Rizzo and Shippy had much broadened the vista of integral equation method for engineering applications.

Cruse went on to write his thesis on boundary integral solutions in elastodynamics and in 1968 published two papers as a result [44,53]. He then left for Carnegie-Mellon University. At Carnegie-Mellon, Cruse was encouraged by Swedlow to work on three-dimensional fracture problems [51]. As a first step, he programmed the integral equation to solve three-dimensional elastostatics problems [45]. In 1970 and 1971, Cruse published boundary integral solutions of three-dimensional fracture problems [46,47]. These were among the first numerical solutions of three-dimensional fracture problems [50], as the first finite difference 3-D fracture solution was done by Ayres [3] in 1970, and the first finite element solution was accomplished by Tracey [170] in 1971.

In 1971 Cruse in his work on elastoplastic flow [163] referred the methods that distributed single- and double-layer potential at fictitious densities, such as those based on the Fredholm integrals and Kupradze's method, as the 'indirect potential methods', and the methods that utilized Green's formality, such as Green's third identity and the Somigliana integral, as the 'direct potential methods'.

However, as Cruse described [51], 'the editor of the IJSS, Professor George Hermann, objected to using such a non-descriptive title as the 'direct potential method'. So, I coined the title boundary-integral equation (BIE) method' [163]. These terms, direct method, indirect method, and boundary integral equation method (BIEM), have become standard terminologies in BEM literature.

In 1973, Cruse resigned from Carnegie-Mellon University and joined Pratt & Whitney Aircraft. He continued to promote the industrial application of the boundary integral equation method. In 1975, Cruse and Rizzo organized the first dedicated boundary integral equation method meeting under the auspices of the Applied Mechanics Division of the American Society of Mechanical Engineers (ASME) in Troy, New York. The proceedings of the meeting [54] reflected the rapid growth of the boundary integral equation method to cover a broad range of applications that included water waves [153], transient phenomena in solids (heat conduction, viscoelasticity, and wave propagation) [155], fracture mechanics [49], elastoplastic problems [119], and rock mechanics [1].

The next international meeting on boundary integral equation method was held in 1977 as the First International Symposium on Innovative Numerical Analysis in Applied Engineering Sciences, at Versailles, France, organized by Cruse and Lachat [52]. This symposium, according to Cruse [51], 'deliberately sought non-FEM papers'. In the same conference, *Carlos Alberto Brebbia* (1948-) was invited to give a keynote address on different mixed formulations for fluid dynamics. Instead, at the last moment, Brebbia decided to talk about applying similar ideas to solve boundary integral equations using 'boundary elements' [23]. In the same year, Jaswon and Symm published the first book on numerical solution of boundary integral equations [97].

9.1. Rizzo

Frank Joseph Rizzo (1938-) was born in Chicago, IL. After graduating form St Rita High School in 1955, he attended the University of Illinois at Chicago. Two years later he transferred to the Urbana campus and received his BS degree in 1960, MS degree in 1961, and PhD in 1964. While pursuing the graduate degrees, he was employed as a half-time teaching staff in the Department of Theoretical and Applied Mechanics. In 1964, he began his career as an Assistant Professor at the University of Washington. Two years later, he left for the University of Kentucky, where he stayed for the next 20 years. In 1987, Rizzo moved to Iowa State University and served as the Head of the Department of Engineering Sciences and Mechanics, which later became a part of the Aerospace Engineering and Engineering Mechanics Department. In late 1989, he returned to his alma mater, the University of Illinois at Urbana-Champaign, to become the Head of the Department of Theoretical and Applied Mechanics. Near the end of 1991, he returned to the Iowa State University and remained there until his retirement in 2000. Rizzo's 1967 article 'An integral equation approach to boundary value problems of classical elastostatics', which was cited more than 300 times as of 2003 based on the *Web of Science* search [195], has much stimulated the modern day development of the boundary integral equation method.

9.2. Cruse

Thomas Allen Cruse (1941-) was born in Anderson, Indiana. After graduation from Riverside Polytechnic High School, Cruse entered Stanford University, where he obtained a BS degree in Mechanical Engineering in 1963, and a MS in Engineering Mechanics in 1964. After a year working with the Boeing Company, he enrolled in 1965 at the University of Washington to pursue a PhD degree, which he was awarded in 1967. In the same year, Cruse joined Carnegie-Mellon University as an Assistant Professor. In 1973 Cruse resigned from Carnegie Mellon and joined Pratt & Whitney Aircraft Group, where he spent the next 10 years. In 1983, he moved to the Southwest Research Institute at San Antonio, Texas, where he stayed until 1990. In that year Cruse returned to the academia by joining the Vanderbilt University as the holder of the H. Fort Flower Professor of Mechanical Engineering. He retired in 1999 as the Associate Dean for Research and Graduate Affairs of the College of Engineering at Vanderbilt University.

10. Boundary element method

While Rizzo in the US was greatly inspired by the work of Jaswon, Ponter, and Symm [93,95,164] at the Imperial College, London, in the early 1960s on potential problems, these efforts went largely unnoticed in the United Kingdom. In the late 1960s, another group in UK started to investigate integral equations. According to Watson [191]: 'I was introduced to boundary integral equations in 1966, as a research student at the University of Southampton under Hugh Tottenham… Tottenham possessed a remarkable library of Soviet books, and encouraged research students to explore possible applications of the work of Muskhelishvili [126], Kupradze [104] and others. I laboured through Muskhelishvili's complex variable theory, and struggled with Kupradze's tortuous mathematical notation without the benefit of being able to read the text.… I pondered among other things the problems posed by edges and corners and came to understand that the fictitious force densities would probably tend to infinity near such features. My attempts to determine the nature of the supposed singularities, however, were unsuccessful.' Doctoral dissertations based on indirect methods of Kupradze produced around this time included that by Banerjee [6] in 1970, and Watson [190] and Tomlin [169] in 1973.

At that time, Brebbia was also a PhD student at the University of Southampton under Tottenham. His research was on the numerical solution of complex double curved shell structures and he investigated a range of techniques including variational methods, finite elements, and integral equations [18]. As a part of his dissertation work, Brebbia spent 18 months at Massachusetts Institute of Technology, first working with *Eric Reissner* (1913–1996), and then Jerry Connor. Reissner's introduction of the mixed formulations of variational principles coupled with Brebbia's knowledge of integral equations gave Brebbia a better understanding of the generalized weak formulations. Jerry Connor was at that time carrying out research in the solution of mixed formulations using finite elements. The collaboration between Brebbia and Connor resulted in two finite element books [24, 39].

With his new knowledge, Brebbia returned to Southampton and completed his thesis on shell analysis [18]. He was appointed a Lecturer at the Civil Engineering Department. He continued his effort in producing integral equation result for complex shell elements, but decided it was insufficiently versatile to warrant further development. It was in 1970 that Brebbia gave his PhD student Jean-Claude Lachat the task of developing boundary integral equation formulation seeking to obtain the same versatility as curved finite elements.

Until that time, the US and the UK schools have been largely working in isolation from each other, except for the influence of Jaswon's work on Rizzo's research, and the fundamental ideas of mixed formulations migrated from MIT to Southampton by Brebbia. In 1972, Brebbia was organizing the First International Conference on Variational Methods in Engineering at Southampton with Tottenham [27]. Although both organizers were working on integral equations at that time, the inclusion of a special session on this subject was an afterthought on the assumption that it would be possible to tie the boundary integral equations with the variational formulation. Brebbia invited Cruse to deliver an opening lecture on the boundary-integral equation method in the conference [48]. As recalled by Brebbia [22]: 'Since the beginning of the 1960s, a group existed at Southampton University working on applications of integral equations in engineering. Our work was up to then directly related to the European Mathematical Schools that originated in Russia. The inclusion of Boundary Integral Equations as one of the topics of the conference was a last moment decision, as up to then they were not interpreted in a variational way… Meeting Tom opened to us the scenario of the research in BIE taking place in America, most of it related to his work. It was a historical landmark for our group and from then on, we continued to collaborate very closely with Tom'.

The above conference was also the occasion for Brebbia, Cruse, and Lachat to get together to discuss the latest research. Lachat went on to offer Watson a position in his laboratory through Brebbia [23]. In Watson's recollection [191]: 'Towards the end of 1971 I was contacted by Brebbia, who asked that I pay him an overnight visit to discuss the implementation of boundary element methods with his external PhD student, Jean-Claude Lachat. … Lachat discussed with me through Brebbia as interpreter some aspects of boundary elements, and then produced a job application form, requesting that I reply to his offer within four weeks. … I decided to accept. Lachat, … was head of the Département Théorique et Engrenages [at Centre Technique des Industries Mécaniques] … At the beginning of 1973 I started work on boundary elements. I was to develop firstly a program for plane strain, then one for three dimensional analysis. Lachat knew of the work of Rizzo and Cruse, and proposed that the direct formulation be used. I readily agreed, having met Cruse in 1972 and discussed with him the direct and indirect approaches. The finite element programming had been most instructive in respect of shape functions, Gaussian quadrature and out-of-core simultaneous equation solution techniques. It seemed clear that the boundary elements should be isoparametric, with at least quadratic variation so that curved surfaces could be modelled accurately. Analytical integration was then out of the question, and Gaussian quadrature was far superior to Simpson's rule. Adaptations of Gaussian quadrature could integrate weakly singular functions, but Cauchy principal values could not be computed directly by quadrature. … The problem of Cauchy principal values was solved by not calculating them'. Lachat finished his dissertation in 1975 [107], and the paper of Lachat and Watson [108] was published in 1976, which was considered as the first published work that incorporates the above-mentioned finite element ideas into boundary integral method.

Up to 1977 the numerical method for solving integral equations had been called the 'boundary integral equation method', following Cruse's naming. However, with the growing popularity of the finite element method, it became clear that many of the finite element ideas can be applied to the numerical technique solving boundary integral equations. This is particularly demonstrated in the work of Lachat and Watson [108]. Furthermore, parallel to the theoretical development of finite element method, it was shown that the weighted residual technique can be used to derive the boundary integral equations [19,25]. The term 'boundary element method,' mirroring 'finite element method,' finally emerged in 1977.

The creation of the term 'boundary element method' was a collective effort by the research group at the University of Southampton. According to José Dominguez [57]: 'The term Boundary Element Method was coined by C.A. Brebbia, J. Dominguez, P. K. Banerjee and R. Butterfield at the University of Southampton. It was used for the first time in three publications of these authors appeared in 1977: a journal paper by Brebbia and Dominguez [25], a [chapter in book] by Banerjee and Butterfield [7], and Dominguez's PhD Thesis [56] (in Spanish). Those four authors never wrote a paper on the subject together but they collectively came to this name at the University of Southampton. At that time, R. Butterfield was a [Senior Lecturer], C.A. Brebbia was a Senior Lecturer, J. Dominguez was a Visiting Research Fellow, and P.K. Banerjee, who had been student and researcher at Southampton [earlier], was a frequent visitor. In some occasions, this group or part of it, met for lunch at the University or even participated at courses organized by Carlos Brebbia in Southampton or London. The phrase Boundary Element Method came out as part of the discussions in one of these meetings and was used by all of them in their immediate work.' (text in square brackets is correction by the authors.) It was also stated in a preface by Brebbia [19]: 'The term 'boundary element' originated within the Department of Civil Engineering at Southampton University. It is used to indicate the method whereby the external surface of a domain is divided into a series of elements over which the functions under consideration can vary in different ways, in much the same manner as in finite elements.'

Brebbia presented the boundary element method using the weighted residuals formulation [19,21,25]. The development of solving boundary value problems using functions defined on local domains with low degree of continuity was strongly influenced by the development of extended variational principles and weighted residuals in the mid 1960s. Key players included Eric Reissner [139] and Kyuichiro Washizu [187], who pioneered the use of mixed variational statements that allowed the flexibility in choosing localized functions. To deal with non-conservative and time-dependent problems, the strategy shifted from the variational approach to the method of weighted residuals combined with the concept of weak forms. Brebbia [19] showed that one could generate a spectrum of methods ranging from finite elements to boundary elements.

Consider a function ϕ satisfying the linear partial differential operator $\mathfrak{L}$ in the following fashion

$$\mathfrak{L}\{\phi\} = b(\mathbf{x}); \quad \mathbf{x} \in \Omega \tag{102}$$

and subject to the essential and natural boundary conditions

$$\begin{aligned} S\{\phi\} &= f(\mathbf{x}); \quad \mathbf{x} \in \Gamma_1 \\ N\{\phi\} &= g(\mathbf{x}); \quad \mathbf{x} \in \Gamma_2 \end{aligned} \tag{103}$$

where S and N are the corresponding differential operators. Our goal is to find the approximate solution that minimizes the error with respect to a weighing function w in the following fashion:

$$\begin{aligned} &\langle \mathfrak{L}\{\phi\} - b, w\rangle_\Omega \\ &\quad = \langle N\{\phi\} - g, S^*\{w\}\rangle_{\Gamma_2} - \langle S\{\phi\} - f, N^*\{w\}\rangle_{\Gamma_1} \end{aligned} \tag{104}$$

where S^* and N^* are the adjoint operators of S and N, and the angle brackets denote the inner product,

$$\langle \alpha, \beta\rangle_\gamma = \int_\gamma \alpha(\mathbf{x})\beta(\mathbf{x})\mathrm{d}\mathbf{x} \tag{105}$$

Eq. (104) can be considered as the theoretical basis for a number of numerical methods [19]. For example, finite difference can be interpreted as a method using Dirac delta function as the weighing function and enforcing the boundary conditions exactly. The well-known Galerkin formulation in finite element method uses of the basis function for w the same as that used for the approximation of ϕ.

For the boundary element formulation, we perform integration by parts on (104) for as many times as necessary to obtain

$$\begin{aligned} \langle \phi, \mathfrak{L}^*\{w\}\rangle_\Omega &= \langle S\{\phi\}, N^*\{w\}\rangle_{\Gamma_2} - \langle N\{\phi\}, S^*\{w\}\rangle_{\Gamma_1} \\ &\quad + \langle f, N^*\{w\}\rangle_{\Gamma_1} - \langle g, S^*\{w\}\rangle_{\Gamma_2} + \langle b, w\rangle_\Omega \end{aligned} \tag{106}$$

where $\mathfrak{L}^*$ is the adjoint operators of $\mathfrak{L}$. The idea for the boundary method is to replace w by the fundamental solution G^*, which satisfies

$$\mathfrak{L}^*\{G^*\} = \delta \tag{107}$$

such that (106) reduces to

$$\begin{aligned} \phi &= \langle S\{\phi\}, N^*\{G^*\}\rangle_{\Gamma_2} - \langle N\{\phi\}, S^*\{G^*\}\rangle_{\Gamma_1} \\ &\quad + \langle f, N^*\{G^*\}\rangle_{\Gamma_1} - \langle g, S^*\{G^*\}\rangle_{\Gamma_2} + \langle b, G^*\rangle_\Omega \end{aligned} \tag{108}$$

This is the weighted residual formulation for boundary element method. For the case of Laplace equation, which is self-adjoint, with the boundary conditions

$$\begin{aligned} \phi &= f(\mathbf{x}); \quad \mathbf{x} \in \Gamma_1 \\ \frac{\partial \phi}{\partial n} &= g(\mathbf{x}); \quad \mathbf{x} \in \Gamma_2 \end{aligned} \tag{109}$$

Eq. (108) becomes

$$\phi = -\frac{1}{4\pi}\iint_{\Gamma_1}\frac{1}{r}\frac{\partial\phi}{\partial n}\,\mathrm{d}S + \frac{1}{4\pi}\iint_{\Gamma_1} f\,\frac{\partial(1/r)}{\partial n}\,\mathrm{d}S - \frac{1}{4\pi}\iint_{\Gamma_2}\frac{1}{r}g\,\mathrm{d}S + \frac{1}{4\pi}\iint_{\Gamma_2}\phi\,\frac{\partial(1/r)}{\partial n}\,\mathrm{d}S \tag{110}$$

which is just Green's formula (25) with boundary conditions substituted in.

In commissioning the term 'boundary element method', Brebbia considered that the method would gain in generality if it was based on the mixed principles and weighted residual formulations. Others have used different interpretations. For example, Banerjee and Butterfield [7] in 1977 followed Kupradze's idea of distributing sources for solving potential problems and forces for elasticity problems, and also called it 'boundary element method.' Similarly, the articles by Brady and Bray [16,17] in 1978 used the term BEM, but were also based on the indirect formulation. Even in Brebbia's 1978 book on boundary element method [19], the indirect formulation was considered, thought not in details.

In the subsequent years, the term BEM has been broadly used as a generic term for a number of boundary based numerical schemes, whether 'elements' were used or not. The present article will not argue whether in certain instances the term boundary element is a misnomer or not. As the present review has demonstrated, the theoretical foundations of these methods are closely related and the histories of their development intertwined. Hence, as indicated in Section 1, the present review accommodates the broadest usage.

In 1978, Brebbia published the first textbook on BEM '*The Boundary Element Method for Engineers*' [19]. The book contained a series of computer codes developed by Dominguez. In the same year, Brebbia organized the first conference dedicated to the BEM: the First International Conference on Boundary Element Methods, at the University of Southampton [20]. This conference series has become an annual event and is nowadays organized by the Wessex Institute of Technology. The most recent in the series as of this writing is the 26th conference in 2004 [26]. In 1984 Brebbia founded the Journal '*Engineering Analysis—Innovations in Computational Techniques*.' It initially published papers involving boundary element as well as other numerical methods. In 1989, the journal was renamed to '*Engineering Analysis with Boundary Elements*' and became a journal dedicated to the boundary element method. The journal is at the present published by Elsevier and enjoys a high impact factor among the engineering science and numerical method journals.

10.1. Brebbia

Carlos Alberto Brebbia (1948-) was born in Rosario, Argentina. He received a BS degree in civil engineering from the University of Litoral, Rosario. He did early research on the application of Volterra equations to creep buckling and other problems. His mentor there was José Nestor Distefano, latterly of the University of California, Berkeley. Brebbia went to the University of Southampton, UK to carry out his PhD study under Hugh Tottenham. During the whole 1966 and first 6 months in 1967, he visited MIT and conducted research under Eric Reissner and Jerry Connor. He attributed his success with FEM as well as BEM to these great teachers. Brebbia was granted his PhD at Southampton in 1967. After a year's research at the UK Electricity Board Laboratories, in 1970 Brebbia started working as a Lecturer at Southampton. In 1975, he accepted a position as Associate Professor at Princeton University, where he stayed for over a year. He then returned to Southampton where he eventually became a Reader. In 1979, Brebbia was again in the US holding a full professor position at the University of California, Irvine. In 1981, he moved back to the UK and founded the Wessex Institute of Technology as an international focus for BEM research. He has been serving as its Director since.

11. Conclusion

In this article we reviewed the heritage and the early history of the boundary element method The heritage is traced to its mathematical foundation developed in the late eighteenth to early 18th century, in terms of the Laplace equation, the existence and uniqueness of solution of boundary value problems, the Gauss and Stokes theorems that allowed the reduction in spatial dimensions, the Green's identities, Green's function, the Fredholm integral equations, and the extension of Green's formula to acoustics, elasticity, and other physical problems. The pioneers behind these mathematical developments were celebrated with short biographies.

We then observe that in the first half of the 20th century there existed various attempts to find numerical solutions without the aid of the modern day electronic computers. Once electronic computers became widely available, 1960s marked the period that 'a hundred flowers bloomed'—all kinds of different ideas were tried. The major development of the modern numerical method, known as the boundary element method, took shape in the 1970s.

With such a rich and complex history, a question like 'who founded the boundary element method and when' cannot be properly answered. First, we may need to qualify the definition of what is the 'boundary element method.' However, it is possible to establish a few important events that provided the critical momentum, which eventually led to the modern day movement. From these events we identify the work done by Jaswon and his co-workers [93,95,164] at the Imperial College in 1963 on the direct and indirect methods for potential problems, and the work by Kupradze and his co-workers [104,105] at Tbilisi State University around 1965 on potential and elasticity problems, as two very important events that had spawned followers. In the US, Jaswon's work inspired Rizzo to develop his 1967 work on the numerical solution of Somigliana integral equation [142]. With followers like Cruse and others, the method, known as the boundary integral equation method at that time, soon prospered. In the UK, a group at the University of Southampton led by Tottenham spent much effort in the early 1970s pursuing Kupradze's methodology. It took Brebbia, who was cross-educated between UK (Southampton) and US (MIT), and Cruse getting together in the mid-1970s to bring cross-fertilization between the two schools, which eventually led to the international movement of boundary element method.

The subsequent conferences and journal organized in the 1980s helped to propel the boundary element method to its mainstream status. By early 1990s, more than 500 journal articles per year were related to this subject. These and the later developments, however, are left to future writers of the BEM history to explore.

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Appendix A. Bibliographic search method

An online search through the bibliographic database *Web of Science* was conducted on May 3, 2004. At the time of the search, the database referred to as the *Science Citation Index Expanded* [195] contained 27 million entries from 5900 major scientific journals covering the period from 1945 to present. A search in the *topic field* based on the combination of key phrases 'boundary element or boundary elements or boundary integral' was conducted. The search matched these phrases in article titles, keywords, and abstracts. These criteria yielded 10,126 articles (Table 1). These entries were further sorted by their publication year and presented as annual number of publication in Fig. 1.

For comparison purposes, two most widely known numerical methods, the finite element method and the finite difference method, were also searched. The combination of key phrases 'finite element or finite elements' and 'finite difference or finite differences', respectively, yielded 66,237 and 19,531 articles. Two less known numerical methods, the finite volume method and the collocation method, were also searched. These results are summarized in Table 1.

The reader should be cautioned that the search method is not precise. The key phrase match is conducted only in the available title, keyword, and abstract fields, and not in the main text. Not all entries in the *Web of Science* database contain an abstract. In fact, most early entries contain neither abstract nor keyword. In those cases, titles alone were searched. This search missed most of the early articles. For example, Rizzo's 1967 classical work [142] '*An integral equation approach to boundary value problems of classical elastostatics*' was missed in the BEM search because none of the above-mentioned key phrases was contained in the title. In fact, all the early boundary integral equation method articles were missed and the first entry was dated 1974 (see Fig. 1). If the search phrase 'integral equation' were used, the Rizzo article would have been found. However, the phrase 'integral equation' was avoided because it would generate many mathematical articles that are not of numerical nature.

While there exist missing articles, we also acknowledge the fact that even if an article was selected based on the matching key phrase such as 'boundary element,' not necessarily the article utilized BEM for numerical solution; hence it may not belong to the category. However, due to the large number of entries involved, we can only rely on automatic search and no attempt was made to adjust the data by conducting article-by-article inspection. Hence the reported result should be regarded as qualitative.

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